Transfinite Cardinals - RIP

Legal Information

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Abstract

The purpose of this document is to prove positively and rigorously that there are no transfinite cardinal numbers, other than $\aleph_0$, by showing that the traditional arguments of their existence are logically invalid, and proving that there is only the one transfinite cardinal number, $\aleph_0$.

The definition of transfinite cardinal numbers depends upon the non-denumerability of the set of real numbers and Cantor's power set theorem. This document proves that the diagonal number "constructed" in the diagonal method, and the subset "constructed" in the power set argument are both self-contradictory under the condition of the antithesis, and consequently, cannot be used for the claimed existence of transfinite cardinal numbers beyond $\aleph_0$.

The document proves, with certainty, that the set of irrational numbers is equivalent to the set of rational numbers, using the irrational number version of Dedekind cuts, with the consequence that transfinite cardinal numbers beyond $\aleph_0$ cannot exist.

Preface

In the year 2007, I published a document on the `https://www.victella.me` website titled **The Collapse of Transfinite Cardinals**. In that document I proved, rigorously, that Cantor's arguments for the non-denumerability of the set of real numbers and the existence of higher transfinite cardinals were logically invalid, but I was still open to the possibility of the existence of those cardinals. I abandoned interest in the subject since 2007, but certain events in 2024 revived my interest.

I considered the notion of partitioning the unit interval into smaller and smaller disjoint sets. It was obvious that, no matter how many times that the unit interval is recursively "split", the number of disjoint sets will always be equal to the number of "splits". This reminded me of Dedekind cuts, which are a kind of "splitting" of the rational number line. So I considered whether I could use the irrational version of Dedekind cuts to prove rigorously that the rational numbers and irrational numbers are equivalent. And, not to my surprise, that was indeed the case.

In this document, not only do I definitely and rigorously prove that Cantor's arguments are invalid, but also prove, using the irrational version of Dedekind cuts, that the higher transfinite cardinals do not exist — that the real numbers and all infinite sets are indeed denumerable.

The ideas in this document have been conceived entirely by me (a retired but non-practising pure mathematician) independently of any other work that may be out there, except, of course, for the standard mathematical definitions and theorems.

Victor Vella

Perth, Western Australia
15 March 2025

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Document Conventions

The following conventions are used in this document.

Document Conventions

Symbol	Meaning
‹x›	separates x as a unit of information from the surrounding text.
$x\cdots$	middle ellipsis means zero, one, or more of the same kind as x.
...	ellipsis represents omitted text (as usual).
text	maroon coloured italic text is a link to the text's definition.
text	underlined green text is a link into the document.

Introduction

A *cardinal number* indicates the number of elements in a finite set. *Cardinal numbers* were generalised by Georg Cantor [1845–1918] to indicate the relative number of elements in *infinite sets* using the notion of *equivalent sets*. Cantor believed, as do modern mathematicians, that the "size" of some *infinite sets* is greater than the size of others. The different "sizes", indicated by the *cardinal numbers* of those sets, are symbolised by $\aleph_0$, $\aleph_1$, $\aleph_2$, and so forth. Those *cardinal numbers* are called *transfinite cardinal numbers* as opposed to the *cardinal numbers* relating to finite sets, which are called *finite cardinal numbers*. However, the basis for believing that *infinite sets* have different *cardinalities* is seriously flawed. Moreover, it can be proven rigorously and conclusively that *infinite sets* do not have different *cardinalities*. The only *transfinite cardinal number* that can be admitted is $\aleph_0$.

It is assumed that the reader has some familiarity with Cantor's diagonal argument, and the argument for Cantor's power set theorem. It is also assumed that the reader is familiar with mathematical logic and has a proper understanding of logical proofs and deductions involving quantifiers. This document is written for mathematicians or readers who are mathematically proficient.

Also, in this document, the term "invalid" used in the context of logical arguments means "not abiding by the proper rules of logic" rather than the flawed and ill-conceived definition that logicians and mathematicians use. "Invalid" in this document does not necessarily mean FALSE.

The first chapter proves in detail that the diagonal argument and the argument for Cantor's power set theorem are invalid. The second chapter proves rigorously that all *infinite sets* are *denumerable*.

1

The Diagonal Argument

1.1 Preliminaries

The principal argument for believing in the higher *transfinite cardinals* is the one based on Cantor's diagonal argument. That sloppy, intuitively-based argument uses Cantor's diagonal method as its principal technique. The diagonal method is invalid, as will be proven in this document using rigorous mathematical logic. Furthermore, again using rigorous mathematical logic, this document will prove conclusively that all *infinite sets* are *equivalent* to the *natural numbers*, therefore proving that there are no more *cardinal numbers* beyond $\aleph_0$.

The problem is with the generalised form of the diagonal method, which claims that it is possible for there to be an entity belonging to a set of all such entities but different from each member of that set — the diagonal method used in the diagonal argument is just a specific case of that general method. But, that general method is obviously self-contradictory, and consequently, so is the diagonal method. Cantor's power set theorem also uses a specific case of that general method, and is therefore also self-contradictory.

It is actually impossible for there to be an entity belonging to a set of all such entities but different from each member of that set. In mathematical logic, such a claim is written as follows.

$\exists r \in S\ (\forall x \in S\ (r \neq x))$ (1)

The statement above says that there is a member, r, belonging to a set, S, such that for each member, x, of the set S (including that member r), that r is different from x. Now, since one of the x's is an r, then $r \neq x$ becomes $r \neq r$ in that case, which is necessarily FALSE (ie: a contradiction) making the statement self-contradictory. That self-contradictory statement is expressed as

Theorem 1: $\exists r \in S\ (\forall x \in S\ (r \neq x)) \Leftrightarrow \perp$

where $\perp$ is the symbol for necessary falsehood. The left side of the theorem is self-contradictory by itself in all cases and in all contexts; there are no exceptions, including when used with the diagonal argument. For example, it is impossible for the same given triangle to be different from each member of the set of all triangles. Theorem 1 is proven in A.1_Proof of Theorem 1.

Two cases relating to Theorem 1 are of interest.

Theorem 2: $(S_2 = S_1) \Rightarrow \neg \exists r \in S_1\ (\forall x \in S_2\ (r \neq x))$

Theorem 2 says that, if two sets, S_1 and S_2, are equal, then it is not the case that there is a member of S_1 that differs from each member of S_2. This theorem is consistent with Theorem 1. Theorem 2 is proven in A.2_Proof of Theorem 2.

Theorem 3: $(S_2 \subset S_1) \Rightarrow \exists r \in S_1\ (\forall x \in S_2\ (r \neq x))$

Theorem 3 says that, if the set S_2 is a *proper subset* of the set S_1, then there is a member of S_1 that differs from all members of S_2. Note that the consequent of Theorem 3 is similar to statement (1), but, in this case, the statement is not self-contradictory. For example, there is at least one rectangle that is different from each member of the set of all squares; the set of all squares being a *proper subset* of the set of all rectangles. Theorem 3 is proven in A.3_Proof of Theorem 3.

1.2 Equivalent Sets

The *cardinality* of a set depends on the notion of *equivalent sets*. Two sets, A and B, are said to be *equivalent*, denoted by $\langle A \sim B\rangle$, if and only if the following statement is TRUE.

$$\exists f\,(f \in \{g : g{:}A \to B\} \wedge \forall y{\in}B\,(\exists x{\in}A\,(f(x) = y)) \wedge \forall x_1, x_2{\in}A\,((x_1 \neq x_2) \to (f(x_1) \neq f(x_2)))) \quad (2)$$

Statement 2 is not as complicated as it appears. It says that there is a function, f, whose domain is A and co-domain is B $(f{:}A \to B)$, where each member, y, of B has a pre-image, x, in A $(\forall y{\in}B\,(\exists x{\in}A\,(f(x) = y)))$, and each member of A maps to a distinct member of B $(\forall x_1, x_2{\in}A\,((x_1 \neq x_2) \to (f(x_1) \neq f(x_2))))$. In other words, there is a one-to-one correspondence between the sets A and B $(A \sim B)$. Now, if there is such a function (ie: if the members of the two sets correspond one-to-one), then those two sets are said to be *equivalent*. It is vitally important to note that, if there is such a function f, then the range of f (**ran** f) is identical to the co-domain of f (**cod** f). That is to say, **ran** f = **cod** f ($\langle$**cod** $f = B\rangle$ by definition), which is a consequence of statement 2. By contraposition, if for all functions f, **ran** $f \neq$ **cod** f, and because $\langle$**ran** $f \subseteq$ **cod** $f\rangle$ by definition, if for all functions f, **ran** $f \subset$ **cod** f, then $A \nsim B$.

In summary: if there is at least one function, $f{:}A \to B$, that is *bijective* then $A \sim B$ and **ran** f = **cod** $f = B$. If for all functions, $f{:}A \to B$, **ran** $f \subset$ **cod** f, then $A \nsim B$.

1.3 Cardinal Numbers

In mathematics, it is useful to define the number of elements in a set (the elements of a set are always distinct). To be useful, the definition needs to be represented by mathematical statements. A *cardinal number* is the family of sets that are *equivalent* to a given set — each member of such a family of sets is associated with the same *cardinal number*. Note that a *cardinal number* is not the same as a *natural number*, but there is a correspondence between *cardinal numbers* and *natural numbers* for finite sets.

The *cardinal number* for a finite set is identical to the *cardinal number* for the initial sequence of positive *natural numbers* that are *equivalent* to that set. For example, the set A, where $A = \{10, 3, 30.7, -22\}$, is *equivalent* to the initial sequence of positive *natural numbers* $\{1, 2, 3, 4\}$ $(A \sim \{1, 2, 3, 4\})$. Therefore, the *cardinal number* of A is the same as the *cardinal number* of $\{1, 2, 3, 4\}$ $(|A| = |\{1, 2, 3, 4\}|)$. Now, the symbol for the *cardinal number* of an initial sequence of positive *natural numbers* is the same as the symbol for the maximum number in that sequence. In our example, the maximum number in $\{1, 2, 3, 4\}$ is 4, so the *cardinal number* of $\{1, 2, 3, 4\}$ is 4. Consequently, the *cardinal number* of A is also 4 $(|A| = 4)$.

It must be noted that the symbol of a *cardinal number*, although it looks like the symbol of a *natural number*, is not a *natural number* — it has its own rules which happen to coincide with those of *natural numbers* for finite sets. It is only for convenience that the symbols are identical. Consequently, the *cardinal number* of a finite set indicates the number of elements in that set. *Cardinal numbers* provide a way to define the number of elements in a set mathematically.

The number of elements in an *infinite set* is undefined in terms of ordinary numbers. However, *cardinal numbers* can be used to indicate that an *infinite set* has a number of elements that is greater than any finite number. The *cardinal number* for the set of *natural numbers* is different than the *cardinal number* for every finite set, and is given the symbol $\aleph_0$ (called "aleph-zero"). Consequently, any set that is *equivalent* to the set of *natural numbers* has the same *cardinal number* $\aleph_0$. *Cardinal numbers* for *infinite sets* are called *transfinite cardinal numbers*.

Are all *infinite sets equivalent* to one another? Georg Cantor [1845–1918], for reasons known only to himself, deluded himself into believing that not all *infinite sets* are *equivalent* using a sloppy children's diagonal argument, and hoodwinked virtually all mathematicians and logicians, and others, into worshipping that same belief. It turns out that it can be rigorously proven (in this document), using mathematical logic, that all *infinite sets* are indeed *equivalent* to one another.

1.4 Critique of the Diagonal Argument

The issue is whether the *unit interval* is *equivalent* to the set of *natural numbers* — whether $\mathbb{N} \sim [0, 1]$. The resolution of this issue determines whether there is more than one *transfinite cardinal number*. If it can be proven that $\mathbb{N} \nsim [0, 1]$, then it can easily be deduced that $\mathbb{N} \nsim \mathbb{R}$, and so $|\mathbb{N}| \neq |\mathbb{R}|$. It would then follow that there is a second *cardinal number* $\aleph_1$ $(= |\mathbb{R}|)$. Using a different theorem, called Cantor's power set theorem (which is also invalid for the same general reason that the diagonal argument is invalid), an infinite sequence of *transfinite cardinal numbers*, $\aleph_0, \aleph_1, \aleph_2, \cdots$, would be deduced to exist. It turns out that the diagonal argument and Cantor's theorem are related such that either both are TRUE or both are FALSE.

Using the definition of *equivalent* sets (as already shown at statement 2),

$$A \sim B =_{\text{df}}$$
$$\exists f\, (f \in \{g : g{:}A \to B\} \wedge \forall y{\in}B\, (\exists x{\in}A\, (f(x) = y)) \wedge$$
$$\forall x_1, x_2{\in}A\, ((x_1 \neq x_2) \to (f(x_1) \neq f(x_2)))), \qquad (3)$$

the *equivalence* of $\mathbb{N}$ and $[0, 1]$ is deduced as follows:

$$\mathbb{N} \sim [0, 1] \stackrel{\text{def}}{=}$$
$$\exists f\, (f \in \{g : g{:}\mathbb{N} \to [0, 1]\} \wedge \forall x{\in}[0, 1]\, (\exists n{\in}\mathbb{N}\, (f(n) = x)) \wedge$$
$$\forall n_1, n_2{\in}\mathbb{N}\, ((n_1 \neq n_2) \to (f(n_1) \neq f(n_2)))). \qquad (4)$$

Cantor's diagonal argument tries to prove that $\mathbb{N} \nsim [0, 1]$ by attempting to use the method of proof by contradiction. In this section, ‹$\mathbb{N} \nsim [0, 1]$› will be called the "thesis", and its negation ($\mathbb{N} \sim [0, 1]$) will be called the "antithesis" in relation to the said proof by contradiction. So statement 4 is the antithesis. Proof by contradiction considers the antithesis, and if there is a contradiction between the antithesis and the ZFC axioms (ZFC axioms of set theory), then those axioms logically imply the thesis ($\mathbb{N} \nsim [0, 1]$), but not the antithesis.

The antithesis, given symbolically as it is without specifying the details of $\mathbb{N}$ or $[0, 1]$, is not contradictory to the ZFC axioms, otherwise statement 4 would be invalid. Therefore, the details of the members of $\mathbb{N}$ and $[0, 1]$, along with statement 4, need to be taken into account to proceed with the argument.

The antithesis (statement 4) implies that there is at least one function, f, for which the range of f, the co-domain of f, and the *unit interval* are all equal ($\mathbf{ran}\, f = \mathbf{cod}\, f = [0, 1]$). See 1.2 Equivalent Sets.

The diagonal argument tries to logically "list" the range of f ($\mathbf{ran}\, f$) of the antithesis, and, using the diagonal method on that *list*, hopes to conclude, by contradiction, that the range of f is different than the co-domain of f ($\mathbf{ran}\, f \neq \mathbf{cod}\, f$) for every f. This would imply that $\mathbb{N} \nsim [0, 1]$. See 1.2 Equivalent Sets.

The *list* ($\mathbf{ran}\, f$) is expressed as the set $\{x_1, x_2, x_3, \ldots\}$ ($\mathbf{ran}\, f = \{x_1, x_2, x_3, \ldots\}$), where $\forall i{\in}\mathbb{N}\, (x_i = f(i))$. Each x_i is unique ($\forall i,j{\in}\mathbb{N}\, ((i \neq j) \to (x_i \neq x_j))$) satisfying the $\forall n_1, n_2{\in}\mathbb{N}\, ((n_1 \neq n_2) \to$

$(f(n_1) \neq f(n_2)))$ part of statement 4. Also, the *list* is equal to the *unit interval* $(\{x_1, x_2, x_3, \ldots\} = [0, 1])$, satisfying the $\forall x \in [0, 1]\ (\exists n \in \mathbb{N}\ (f(n) = x))$ part of statement 4.

Referring to statement 4, the claim in the diagonal method is that there exists a number, r (the so-called diagonal number), belonging to the co-domain of f (**cod** f) but is different from each member of the range of f (**ran** f). In mathematical logic, this is generally expressed as

$$\exists r \in (\mathbf{cod}\ f)\ (\forall x \in (\mathbf{ran}\ f)\ (r \neq x)). \qquad (5)$$

However, with the implication of the antithesis, substituting ‹**ran** f = **cod** $f = \{x_1, x_2, \ldots\} = [0, 1]$› directly into statement 5 results in

$$\exists r \in [0, 1]\ (\forall i \in \mathbb{N} \wedge x_i \in [0, 1]\ (r \neq x_i)). \qquad (6)$$

But, STATEMENT 6 IS SELF-CONTRADICTORY as proven by Theorem 1. So, even before the details of the diagonal method begin, the very principle upon which that method rests is flawed — the diagonal method is merely a particular case of statement 6.

Note that the use of the set $\{x_1, x_2, x_3, \ldots\}$ is redundant, but introduced to relate the discussion to the way that the diagonal argument is generally presented. Substituting ‹**ran** f = **cod** $f = [0, 1]$› into statement 5 results in $\exists r \in [0, 1]\ (\forall x \in [0, 1]\ (r \neq x))$, which is self-contradictory as before.

The above ought to be enough to convince rational mathematicians to dismiss the diagonal argument out of existence. But for the sake of the die-hard mathematician (who is typically convinced more by popularity and emotion than by logic) we will press on.

Some mathematicians may have an issue with substituting ‹**ran** f = **cod** $f = [0, 1]$› into statement 5. The antithesis $(\mathbb{N} \sim [0, 1])$ certainly implies ‹**ran** f = **cod** $f = [0, 1]$›, and r is certainly a member of $[0, 1]$ $(r \in [0, 1])$ as defined by the diagonal method. The "list" in the diagonal argument is certainly equal to **ran** f, because that is the part that varies with different functions f of statement 4.

So, we have

$$(\mathbb{N} \sim [0, 1]) \Rightarrow (\mathbf{ran}\ f = [0, 1]). \qquad (7)$$

But, the consequent of logical implication 7 is a particular case of the antecedent of Theorem 2, therefore,

$$(\mathbf{ran}\ f = [0, 1]) \Rightarrow \neg\exists r \in [0, 1]\ (\forall x \in (\mathbf{ran}\ f)\ (r \neq x)). \qquad (8)$$

Putting logical implications 7 and 8 together, we get

$$(\mathbb{N} \sim [0, 1]) \Rightarrow (\mathbf{ran}\ f = [0, 1]) \Rightarrow \neg\exists r \in [0, 1]\ (\forall x \in (\mathbf{ran}\ f)\ (r \neq x)). \qquad (9)$$

So, the antithesis implies that there does <u>not</u> exist a number r (the diagonal number) that is different from each member of the *list* (**ran** f). In plain English, this means that no such diagonal number can be "constructed", "defined", etc. Therefore, in plain English, the diagonal method DOES NOT guarantee that the diagonal number exists. In fact, the diagonal method DOES guarantee that there is NO such diagonal number at all under the condition (**ran** $f = [0, 1]$) of the antithesis, consequently invalidating the diagonal argument altogether.

But the die-hard mathematician is still not convinced, insisting that

$$\exists r \in [0, 1]\ (\forall x \in (\mathbf{ran}\ f)\ (r \neq x)) \qquad (10)$$

is <u>unconditionally</u> TRUE (because Cantor said so), thereby indicating a contradiction with the consequent of logical implication 9, claiming that the contradiction required by the proof by contradiction has been fulfilled.

The VITAL question for the die-hard mathematician is: what justifies the claim that statement 10 is unconditionally TRUE (ie: that the diagonal method guarantees that the diagonal number r exists)? Statement 10 is certainly <u>not</u> a tautology — Theorem 1 sees to that. It is not deduced from the ZFC axioms since the truth of the statement is conditional upon ‹**ran** $f \subset [0, 1]$› (Theorem 3). The answer can only be that it is an arbitrarily introduced statement dogmatically asserted to be unconditionally TRUE (for example, by the word "construct"). But, it is forbidden to <u>arbitrarily</u> introduce statements in a logical deduction, otherwise anything can be proven.

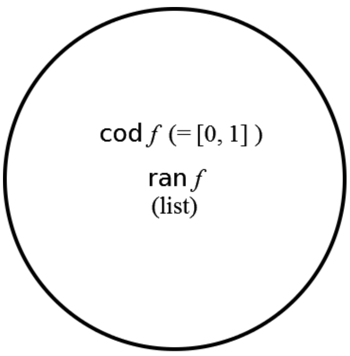
However, there <u>is</u> a condition under which statement 10 is TRUE. After substituting ‹**ran** f› for S_1, and [0, 1] for S_2, Theorem 3 implies that

$$(\textbf{ran}\, f \subset [0, 1]) \Rightarrow \exists r{\in}[0, 1]\ (\forall x{\in}(\textbf{ran}\, f)\ (r \neq x)). \qquad (11)$$

Statement 11 says that, <u>if</u> ‹**ran** f› is a *proper subset* of [0, 1] (= **cod** f), then, indeed, there is a number, r (the diagonal number), that is different from each element of ‹**ran** f› (the *list*). However, this is a natural consequence of *proper subsets*, and statement 11 is not relevant to the antithesis and the diagonal argument. The statement certainly does not prove that it is <u>impossible</u> that ‹**ran** f = [0, 1]›, which is what is required to be proven to claim that no such function f exists satisfying statement 4.

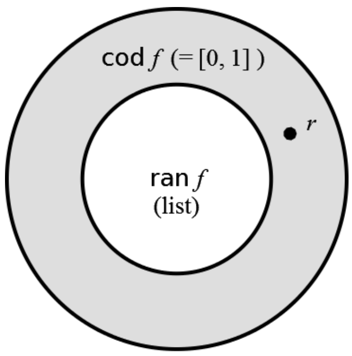
So, in summary to the die-hard mathematician, statement 10 is TRUE <u>only</u> under the <u>precondition</u> that ‹**ran** f› is a *proper subset* of [0, 1] (**ran** $f \subset [0, 1]$), which is an irrelevant condition to the argument — the statement does not <u>prove</u> that ‹**ran** f› <u>must</u> be a *proper subset* of [0, 1]. So, the diagonal argument tacitly <u>presupposes</u> (not deduces) logically that the *list* is a *proper subset* of [0, 1], despite the verbal utterance that the *list* is (hypothetically) equal to [0, 1]. The mistake that the die-hard mathematicians have made is to have assumed that statement 10 is necessarily or axiomatically TRUE, rather than TRUE only when ‹**ran** $f \subset [0, 1]$›.

The relation between r, ‹**ran** f›, and ‹**cod** f› can be summarised using Venn diagrams.



ran f = **cod** f = [0, 1]

r (the diagonal number) does not exist, therefore cannot be used in the diagonal argument.



ran $f \subset [0, 1]$

r (the diagonal number) exists only in the shaded area. This case is irrelevant to the diagonal argument.

The antithesis and the proof by contradiction of the diagonal argument logically involve only the case on the left, making the argument invalid; the case on the right is irrelevant to the argument. But, by dogmatically asserting that statement 10 is TRUE, mathematicians have logically, but unknowingly, considered only the case on the right. However, they ignorantly and naively thought that they were considering the case on the left, mistakenly thinking that a contradiction arose to reject that case — mathematicians have conflated the two cases. The case on the left has not been disproven; it is still possible.

But the fanatical die-hard "mathematician" will still insist that it is the details of the actual diagonal method that proves it unconditionally TRUE (as though an instance of a theorem can, somehow, refute the theorem). For such die-hards, the actual diagonal method will be analysed in the next section.

1.4.1 Details for Stubborn Mathematicians

It has already been shown above that the principle of the diagonal argument is flawed. That principle is that it is possible for there to be an entity belonging to a set of all such entities but different from each member of that set. In the case of the diagonal argument, that translates to: it is possible for there to be a number, r, in the *unit interval* ([0, 1]) belonging to a list (the range of a function f) of all such numbers but different from each member of that list. Keep in mind that the diagonal argument assumes, hypothetically, that the range of f is identical to the *unit interval*.

We now consider the details of the diagonal method in terms of mathematical logic. The diagonal number r can be represented by a sequence of digits as

$$r = \langle a_0, a_1, a_2, \ldots\rangle, \qquad (12)$$

and the function f which maps the *natural numbers* to the *unit interval*, not necessarily as a one-to-one correspondence, can be defined by digit sequences as

$$\forall i \in \mathbb{N}\ (f(i) = x_i = \langle b_{i0}, b_{i1}, b_{i2}, \ldots\rangle), \qquad (13)$$

where $\forall i, j \in \mathbb{N}\ (a_i, b_{i,j} \in \{0, \ldots, 9\})$. The sequences represent the decimal digits of numbers in the *unit interval*. We also have the following equations based on statements 12 and 13 above,

$$L = \mathbf{ran}\, f = \{f(0), f(1), \ldots\} = \{x_0, x_1, \ldots\} = \{\langle b_{00}, b_{01}, \ldots\rangle, \langle b_{10}, b_{11}, \ldots\rangle, \ldots\}. \qquad (14)$$

L represents the "list" that is typically mentioned in the traditional diagonal argument.

We also have the following Lemma deduced from the equations 12 and 14.

Lemma L1: $\forall i \in \mathbb{N} \wedge a_i \in r \wedge b_{i,i} \in x_i\ (a_i \neq b_{i,i}) \Rightarrow \forall i \in \mathbb{N} \wedge x_i \in L\ (r \neq x_i)$.

We now prove that the diagonal method is FALSE if the antithesis is assumed. First, we have that

$$\exists r \in [0, 1]\ (\forall i \in \mathbb{N} \wedge a_i \in r \wedge b_{i,i} \in x_i\ (a_i \neq b_{i,i})) \Rightarrow \text{[DIAGONAL METHOD]}$$
$$\exists r \in [0, 1]\ (\forall i \in \mathbb{N} \wedge x_i \in L\ (r \neq x_i)). \text{ [by Lemma L1]} \qquad (15)$$

But (if the antithesis is now assumed, ie, $\langle L = [0, 1]\rangle$),

$$(L = [0, 1]) \Rightarrow \neg\exists r \in [0, 1]\ (\forall i \in \mathbb{N} \wedge x_i \in L\ (r \neq x_i)). \text{ [by Theorem 2]} \qquad (16)$$

The contrapositive of logical implication 15 is

$$\neg\exists r \in [0, 1]\ (\forall i \in \mathbb{N} \wedge x_i \in L\ (r \neq x_i)) \Rightarrow \neg\exists r \in [0, 1]\ (\forall i \in \mathbb{N} \wedge a_i \in r \wedge b_{i,i} \in x_i\ (a_i \neq b_{i,i})). \quad (17)$$

Therefore, combining logical implications 16 and 17, we get

$$(L = [0, 1]) \Rightarrow \neg\exists r \in [0, 1]\ (\forall i \in \mathbb{N} \wedge a_i \in r \wedge b_{i,i} \in x_i\ (a_i \neq b_{i,i}))$$

and the diagonal method is FALSE if the antithesis is assumed. Note carefully that an external general theorem (Theorem 2) was used in the proof above to prove that the diagonal method is FALSE. Therefore, it is logically invalid to claim that the diagonal method guarantees that r is different from each number in the list L, because r does not exist. All that the diagonal method guarantees is that it is FALSE if the antithesis ($L = [0, 1]$) is assumed.

But mathematical logic is beyond the comprehension of the die-hard mathematician, so the actual presentation of Cantor's diagonal method will be analysed to reveal its flaw.

The following is a demonstration of exactly where the contradiction in the diagonal method arises. The contradiction occurs entirely within the method itself, not as a contradiction with the ZFC axioms.

Below is an illustrative example of the 'list' so popular with that children's diagonal argument.

x_0 = 0.3 . . . .

x_1 = 0. . 7 . . . . . .

x_2 = 0. . . 5 . . . . .

$\vdots \quad \ddots$

x_n = 0.486 . . ? . . . (the diagonal number, r)

$\vdots \quad \ddots$

First of all, in the traditional presentation of the diagonal argument, the variable r, representing the diagonal number, is presented outside the *list*. To be consistent with the antithesis, the variable r MUST be presented IN the actual *list* (as shown in the example above at x_n), not somewhere outside the *list*. Having the variable <u>not</u> in the actual *list* implies a <u>different list</u> than the one assumed by the antithesis. In other words, using a <u>separate</u> variable for the diagonal number implies that the number may be equal to some number in the *list*, or that it may be different from all the numbers in the *list*. However, by the antithesis, the number is in the *unit interval* (which is equal to the *list*), so the diagonal number r should <u>not</u> be presented as a separate variable but presented as one of the numbers <u>in</u> the *list* (ie: x_n). Conversely, if the diagonal number r is different from <u>all</u> of the numbers in the *list*, then the *list* is not the antithesis *list*, but a *proper subset* of the antithesis *list*, even before the diagonal method begins. Therefore, in that case, the diagonal method itself is redundant because it is already presupposed that there is a number, r, not in a *proper subset* of the antithesis *list*, and so no contradiction occurs.

Let us proceed. Take r (= x_n) to be <u>in</u> the *list* since the *list* contains <u>all</u> the *real numbers* (including the diagonal number) in the interval [0, 1] as per the antithesis. The diagonal number is defined by having its k^{th} digit one greater than the k^{th} digit of x_k, except when the k^{th} digit is 9. In that case, the k^{th} digit of x_k, will be 0. With the example *list* above, the 4 is one greater than the 3; the 8 is one great than the 7; the 6 is one greater than the 5; and so on. What digit should replace the question mark (?)? By the very definition of the diagonal method itself, that digit would have to be one greater than itself, which is logically impossible.

The diagonal method is attempting to use the diagonal number itself as part of its own definition in a self-contradictory way; therefore the diagonal method is self-contradictory. No such entity defined in the said manner can exist. If someone wants to claim that the said contradiction implies that the diagonal number r does exist but <u>cannot</u> be in the *list*, then they must accept the logical presupposition in the diagonal method that "the list" is a *proper subset* ($[0, 1]\backslash\{r\}$) of the antithesis *list*, and not the *list* that is actually assumed by the antithesis ([0, 1]) — the antithesis *list* is still possible (with dire consequences for the diagonal argument).

What mathematicians seemed to have done is that they initially assumed that r may or may not be in the *list* (via the antithesis), and then imagined, by naively conflating both cases together, that they had deduced, by the diagonal method, that the diagonal number could not possibly be in the *list*. Here, they failed to separate the two cases of the number r being in the *list* and not in the *list* — the first case is self-contradictory; the second case tacitly presupposes that the presented *list* is a *proper subset* of the antithesis *list* even before the introduction of the diagonal method. Presenting a *proper subset* of the antithesis *list* is irrelevant to the diagonal argument; the existence of a *proper subset* of the antithesis *list* does not logically invalidate the existence of the antithesis *list* itself. In both cases, the diagonal argument fails.

Some mathematicians may want to claim that the antithesis is actually that the function f is a *bijection* (a one-to-one and onto function). They then show that the diagonal method is TRUE only for a *proper subset* of the antithesis function's co-domain, therefore the antithesis function cannot be a *bijection*, claiming that the *bijectivity* of the antithesis is what has been contradicted. Here, they make the following mistake. By claiming that the diagonal method implies a *proper subset* of the antithesis function's co-domain, they have also tacitly presupposed a different function, h (a one-to-one and into function), rather than the function f of the antithesis. So, no contradiction has been obtained with the original antithesis function f. In other words, the diagonal method can only be TRUE with a different function, h, rather than with f (the diagonal method is impossible with f). So all they prove is that the diagonal method, if TRUE, entails that there exists a function, h, that is one-to-one and into. They completely fail to prove that there cannot exist a function f (one-to-one and onto). Here, they conflate the functions h and f. The existence of the function h does not logically invalidate the existence of the function f. This is the sort of confusion that happens when amateurs use pictorial children's methods with subjective terms as proofs in place of proper logical proofs with proper logical terms.

1.5 Critique of Cantor's Power Set Theorem

Cantor's power set theorem (usually just called "Cantor's Theorem") claims that the *cardinality* of any set is strictly less than the *cardinality* of its *power set* ($\forall A\ (|A| < |\wp(A)|)$). For finite sets, this is certainly TRUE, but will not be proven in this document. The issue arises with *infinite sets*; does the strict inequality hold for them? If $A \sim \wp(A)$ is proven for *infinite sets*, then the *cardinality* of an *infinite set* is equal to the *cardinality* of its *power set* ($|A| = |\wp(A)|$), and for all sets, we would have $\forall A\ (|A| \leq |\wp(A)|)$.

The argument used in Cantor's power set theorem will be called the "power set argument" in this document. Just as the diagonal argument is invalid, so too the power set argument is invalid for the same general reason.

The aim of this section is to show that the argument that claims the non-*equivalence* of an *infinite set*, A, and its *power set*, $\wp(A)$, is invalid. The *equivalence* of A and $\wp(A)$ is defined as follows:

$$\begin{aligned} &A \sim \wp(A) \stackrel{\text{def}}{=} \\ &\exists f\,(f \in \{g : g{:}A \to \wp(A)\} \wedge \forall P{\in}\wp(A)\ (\exists x{\in}A\ (f(x) = P)) \wedge \\ &\forall x_1, x_2{\in}A\ ((x_1 \neq x_2) \to (f(x_1) \neq f(x_2)))). \qquad (18) \end{aligned}$$

The power set argument tries to prove that $A \nsim \wp(A)$ by attempting to use the method of proof by contradiction. In this section, $A \nsim \wp(A)$ will be called the "thesis", and its negation ($A \sim \wp(A)$) will be called the "antithesis" in relation to the said proof by contradiction. Proof by contradiction considers the antithesis, and if there is a contradiction between the antithesis and the ZFC axioms (ZFC axioms of set theory), then those axioms logically imply the thesis ($A \nsim \wp(A)$), but not the antithesis.

The antithesis (statement 18) implies that there is at least one function, f, for which the range of f, the co-domain of f, and the *power set* of A are all equal (**ran** f = **cod** $f = \wp(A)$). See 1.2 Equivalent Sets.

Referring to statement 18, the claim in the power set argument is that, for every f, there exists a set, S, belonging to the co-domain of f (**cod** f) but is different from each member of the range of f

(**ran** f). This means that the range of f is different than the co-domain of f (**ran** $f \neq$ **cod** f) for every f, implying that $A \nsim \wp(A)$, and, with a bit more logic, that $\forall A\ (|A| < |\wp(A)|)$.

Just as for the diagonal method, the problem is that the power set argument attempts to achieve the logically impossible in a devious way.

The claim effectively states that the following statement is TRUE.

$\exists S \in (\mathbf{cod}\ f)\ (\forall P \in (\mathbf{ran}\ f)\ (S \neq P)).$ (19)

However, with the implication of the antithesis, substituting ‹**ran** $f =$ **cod** $f = \wp(A)$› directly into statement 19 results in

$\exists S \in \wp(A)\ (\forall P \in \wp(A)\ (S \neq P)).$ (20)

This means that there is a set, S, that is a *subset* of the set A, that is different from each *subset* in the range of f which contains all the *subsets* of A.

But, STATEMENT 20 IS SELF-CONTRADICTORY as proven by Theorem 1. So, even before the details of the power set argument begin, the very principle upon which that argument rests is flawed — the power set argument is merely a particular case of statement 20.

From the antithesis, we have

$(A \sim \wp(A)) \Rightarrow (\mathbf{ran}\ f = \wp(A)).$ (21)

But, the consequent of logical implication 21 is a particular case of the antecedent of Theorem 2, therefore,

$(\mathbf{ran}\ f = \wp(A)) \Rightarrow \neg\exists S \in \wp(A)\ (\forall P \in (\mathbf{ran}\ f)\ (S \neq P)).$ (22)

Putting logical implications 21 and 22 together, we get

$(A \sim \wp(A)) \Rightarrow (\mathbf{ran}\ f = \wp(A)) \Rightarrow \neg\exists S \in \wp(A)\ (\forall P \in (\mathbf{ran}\ f)\ (S \neq P)).$ (23)

So, the antithesis implies that there does not exist a *subset* S of A that is different from each member of the range of f (**ran** f). In plain English, this means that no such *subset* can be "constructed", "defined", etc. Therefore, in plain English, the power set argument DOES NOT guarantee that the *subset* S exists. In fact, the power set argument DOES guarantee that there is NO such *subset* at all under the condition (**ran** $f = \wp(A)$) of the antithesis, consequently invalidating the power set argument altogether. (The reader may notice some *déjà vu* happening here with the diagonal argument.)

But the die-hard mathematician is still not convinced, insisting that

$\exists S \in \wp(A)\ (\forall P \in (\mathbf{ran}\ f)\ (S \neq P))$ (24)

is unconditionally TRUE (because Cantor said so), thereby indicating a contradiction with the consequent of logical implication 23, claiming that the contradiction required by the proof by contradiction has been fulfilled.

The VITAL question for the die-hard mathematician is: what justifies the claim that statement 24 is unconditionally TRUE? Statement 24 is certainly not a tautology — Theorem 1 sees to that. It is not deduced from the ZFC axioms since the truth of the statement is conditional upon ‹**ran** $f \subset \wp(A)$› (Theorem 3). The answer can only be that it is an arbitrarily introduced statement dogmatically asserted to be unconditionally TRUE. But, it is forbidden to arbitrarily introduce statements in a logical deduction, otherwise anything can be proven. (More *déjà vu*.)

However, there is a condition under which statement 24 is TRUE. After substituting ‹**ran** f› for S_1, and $\wp(A)$ for S_2, Theorem 3 implies that

$$(\mathbf{ran}\, f \subset \wp(A)) \Rightarrow \exists S{\in}\wp(A)\ (\forall P{\in}(\mathbf{ran}\, f)\ (S \neq P)). \qquad (25)$$

Statement 25 says that, if ‹**ran** f› is a *proper subset* of $\wp(A)$ (= **cod** f), then, indeed, there is a *subset*, S, that is different from each element of ‹**ran** f›. However, this is a natural consequence of *proper subsets*, and statement 25 is not relevant to the antithesis and the power set argument. The statement certainly does not prove that it is impossible that ‹**ran** $f = \wp(A)$›, which is what is required to be proven to claim that no such function f exists satisfying statement 18.

So, in summary to the die-hard mathematician, statement 24 is TRUE only under the precondition that ‹**ran** f› is a *proper subset* of $\wp(A)$ (**ran** $f \subset \wp(A)$), which is an irrelevant condition to the argument — the statement does not prove that ‹**ran** f› must be a *proper subset* of $\wp(A)$. So, the power set argument tacitly presupposes (not deduces) logically that the range of f is a *proper subset* of $\wp(A)$, despite the verbal utterance that the range of f is (hypothetically) equal to $\wp(A)$. The mistake that the die-hard mathematicians have made is to have assumed that statement 24 is necessarily or axiomatically TRUE, rather than TRUE only when ‹**ran** $f \subset \wp(A)$›.

1.5.1 Details for Stubborn Mathematicians

What is this *subset*, S, magically created by mathematicians, that feigns the conclusion that the *power set* of an *infinite set* has greater *cardinality* than the set?

The magical set, S, is defined this way.

$$S = \{x \in A : x \notin f(x)\} \qquad (26)$$

This set is arrogantly ASSUMED to exist unconditionally by mathematicians. Merely defining a set does not automatically guarantee that it exists (mathematically) because a definition can be self-contradictory or conditional. The correct way to define the set in mathematical logic is

$$\exists S\ (\forall x\ (x \in S \leftrightarrow x \in A \wedge x \notin f(x)) \qquad (27)$$

Statement 27 must be proven to be TRUE, not just assumed to be so. The statement involves the function f, and is therefore conditional (because the existence of f is conditional). If the statement is conditional, then it could be FALSE. And, if it could be FALSE, then it is not derived from the ZFC axioms unconditionally.

Now, the power set argument does deduce a contradiction from statement 27 with the antithesis. But that contradiction is the result of the definition of S, together with the antithesis, being self-contradictory, rather than from a conjunction of a deduction from the statement (and the antithesis) with another statement derived from the ZFC axioms independently of the antithesis. In other words, the contradiction required by the proof by contradiction is a contradiction between **(1)** the antithesis and **(2)** the ZFC axioms entirely independent of the antithesis. Neither of those two lines of logic are permitted to result in a contradiction themselves — the required contradiction needs to be from the conjunction of the conclusions of those two lines. There is no such contradiction between those two lines of logic in the power set argument. The power set argument has only one line of logic that depends on the antithesis, and the resulting contradiction makes that line itself self-contradictory.

The following shows that statement 27 leads to a contradiction using proper mathematical logic. Firstly, we will assume that S is not the null set ($S \neq \varnothing$) to avoid complications. It is unlikely that mathematicians intended that the set be null, so the non-null set assumption is a fair one. In any case, it does not alter the outcome.

$\forall x\ (x \in S \leftrightarrow x \in A \wedge x \notin f(x)) \Rightarrow$ [from definition of S (statement 27)]

$\forall x\ (x \in S \rightarrow x \in A) \Leftrightarrow$ [deduction]

$S \subseteq A \Leftrightarrow$ [by definition of subset]

$S \in \wp(A)$ [by definition of power set] □

$\forall P{\in}\wp(A)\ (\exists x{\in}A\ (f(x) = P)) \Leftrightarrow$ [by definition of the 'onto' part of f]

$\forall P{\in}\{S\}\ (\exists x_0{\in}A\ (f(x_0) = P)) \wedge \forall P{\in}\wp(A)\backslash S\ (\exists x{\in}A\ (f(x) = P)) \Rightarrow$ [separation of S from $\wp(A)$]

$\forall P{\in}\{S\}\ (\exists x_0{\in}A\ (f(x_0) = P)) \Leftrightarrow$ [deduction]

$\exists x_0{\in}A\ (f(x_0) = S) \Leftrightarrow$ [simplification ($P = S$)]

$\exists x_0\ (x_0 \in A \wedge (f(x_0) = S)) \Rightarrow$ [equivalence to the previous statement]

$x_0 \in A$ [interpretation then deduction]

$f(x_0) = S$ [interpretation then deduction] □

$\forall x\ (x \in S \leftrightarrow x \in A \wedge x \notin f(x)) \Leftrightarrow$ [from definition of S (statement 27)]

$\forall x{\in}\{x : x = x_0\}\ (x \in S \leftrightarrow x \in A \wedge x \notin f(x)) \wedge$
$\quad \forall x{\in}\{x : x \neq x_0\}\ (x \in S \leftrightarrow x \in A \wedge x \notin f(x)) \Rightarrow$ [separation of x_0 from x]

$\forall x{\in}\{x : x = x_0\}\ (x \in S \leftrightarrow x \in A \wedge x \notin f(x)) \Leftrightarrow$ [deduction]

$x_0 \in S \leftrightarrow x_0 \in A \wedge x_0 \notin f(x_0) \Leftrightarrow$ [simplification ($x = x_0$)]

$x_0 \in S \leftrightarrow x_0 \in A \wedge x_0 \notin S \Leftrightarrow$ [substitution from $f(x_0) = S$ of a previous step]

$x_0 \notin S \wedge x_0 \notin A \Rightarrow$ [simplification of bi-conditional ($\leftrightarrow$)]

$x_0 \notin A$. [deduction]

$x_0 \notin A \wedge x_0 \in A \Leftrightarrow \perp$ [$x_0 \in A$ from a previous step] ■

Therefore: $\exists S(\forall x\ (x \in S \leftrightarrow x \in A \wedge x \notin f(x))) \Rightarrow \exists S(\perp) \Rightarrow \perp$. [from the deduction above]

Therefore: $\exists S(\forall x\ (x \in S \leftrightarrow x \in A \wedge x \notin f(x))) \Leftrightarrow \perp$. [statement 27 is self-contradictory]

Notice that the argument presented above, showing that S is self-contradictory, is the same as the traditional power set argument, but is a direct proof. If the traditional power set argument is considered valid by mathematicians, then so too the (almost) identical argument above ought to be considered valid. The difference is that, in the traditional power set argument, S is arbitrarily assumed to exist unconditionally without proof (of which there is none) — it is that assumption that makes the traditional power set argument invalid.

So, the contradiction resulting from the power set argument is a consequence of the definition of S being self-contradictory, not a consequence of ‹$A \sim \wp(A)$› being contradictory with the ZFC axioms. A way to avoid the contradiction is to define S as ‹$S = \{x \in A : x \notin f(x) \setminus S\}$›, rather than as the deliberately contrived self-contradictory set S used in the power set argument. Defining S in the said manner allows ‹$x_0 \in A$› without contradiction (allowing f to be 'onto'), but is useless to the power set argument.

It is important to note that the self-contradiction of statement 27 arises only in conjunction with the implied 'onto' function f of the antithesis; no contradiction results for functions, f, that are 'into' (ie: where **ran** $f \subset \wp(A)$). However, the antithesis together with the proof by contradiction of the traditional power set argument logically considers only 'onto' functions, making that argument invalid (as shown above); 'into' functions are irrelevant to the argument. But, by dogmatically asserting that statement 27 is unconditionally TRUE, mathematicians have logically, but unknowingly, considered only 'into' functions. However, they ignorantly and naively thought that they were considering 'onto' functions, mistakenly thinking that a contradiction arose to reject that case — the case where the function f is 'onto' has not been disproven; it is still possible. Mathematicians have conflated the two cases.

In summary, if the antithesis is TRUE then statement 27 (and 26) is self-contradictory (as proven above), and the power set argument is invalid. But mathematicians have dogmatically asserted that the statement is unconditionally TRUE (without justification), and falsely claimed that the resulting contradiction proves that the antithesis is FALSE (ie: that it is impossible that there exists an 'onto' function f). Statement 27 cannot be derived from the ZFC axioms because its truth is conditional on the antithesis — the ZFC axioms are not conditional. So, under the hypothetical assumption that $A \sim \wp(A)$, the traditional power set argument fails.

But the die-hard mathematician is so stubborn (and hard to teach) that further convincing is required. From statement 27 we have

$$\exists S\,(\forall x\,(x \in S \leftrightarrow x \in A \wedge x \notin f(x)) \Rightarrow \exists S{\in}\wp(A)\,(\forall P{\in}(\mathbf{ran}\, f)\,(S \neq P)). \qquad (28)$$

The consequent of logical implication 28 states that the set S is a *subset* of A, and that it is different than any of the *subsets* of A that are in the range of f. Note that no assumption that ‹**ran** $f = \wp(A)$› has been made in statement 28. (Note that **cod** $f = \wp(A)$ by definition.) We now assume the antithesis to obtain

$$(\mathbf{ran}\, f = \wp(A)) \Rightarrow \neg\exists S{\in}\wp(A)\,(\forall P{\in}(\mathbf{ran}\, f)\,(S \neq P)). \text{ [by Theorem 2]} \qquad (29)$$

The contrapositive of logical implication 28 is

$$\neg\exists S{\in}\wp(A)\,(\forall P{\in}(\mathbf{ran}\, f)\,(S \neq P)) \Rightarrow \neg\exists S\,(\forall x\,(x \in S \leftrightarrow x \in A \wedge x \notin f(x)). \qquad (30)$$

Therefore, combining logical implications 29 and 30, we get

$$(\mathbf{ran}\, f = \wp(A)) \Rightarrow \neg\exists S\,(\forall x\,(x \in S \leftrightarrow x \in A \wedge x \notin f(x))$$

and the existence of the magical set, S, assumed in the traditional power set argument to unconditionally exist, is FALSE if the antithesis is assumed (**ran** $f = \wp(A)$). Note carefully that an external general theorem (Theorem 2) was used in the proof above to prove that the existence of S is FALSE. Therefore, it is logically invalid to claim in the power set argument that the set S exists. No fancy words like "construct the *subset* S" or "consider the *subset* S" or "define the *subset* S" is going to make a non-existent set existent (mathematically speaking) unless the set in question is magical and some sort of mathematical wizard brings it into existence (maybe the Wizard of Oz can do it; or maybe Cantor was actually the Wizard of Oz pretending to be a mathematician!).

2

Proof that the Reals are Denumerable

2.1 Preliminaries

It is not possible to determining whether two *infinite sets* are not *equivalent* if the elements of the two sets are regarded as unrelated indivisible units. In principle, it appears that some intrinsic relation between the elements of the two sets needs to be known to determine that they are not *equivalent*. The default situation is that two *infinite sets* are *equivalent* (there is no reason to consider otherwise). However, if it is proposed that any two particular *infinite sets* are not *equivalent*, as is assumed with the *natural numbers* and the *real numbers*, then that proposal may be contested by considering the intrinsic nature of the elements of the two sets.

Mathematicians currently (in 2025) assume that the rational numbers are not *equivalent* to the irrational numbers. However, by examining the intrinsic nature of both sets of numbers, it can be rigorously proven that the two sets are indeed *equivalent*. The rational numbers and the irrational numbers can both be defined within a single system called Dedekind cuts. It turns out that the definition of Dedekind cuts guarantees that the *cardinality* of the irrational numbers is identical to the *cardinality* of the rational numbers. Consequently, the *real numbers* are not *denumerable*, contrary to current mathematical "wisdom".

The definition of a Dedekind cut for an irrational number is an ordered pair (L, R) of *subsets* of the rational numbers, $\mathbb{Q}$, into two non-null sets, L and R, satisfying the following conditions:

1. $L \cup R = \mathbb{Q}$ (L and R together contain all the rational numbers)

2. $L \cap R = \varnothing$ (L and R are disjoint; they have no common elements)

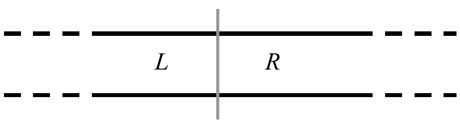
3. $\forall a{\in}L\ (\forall b{\in}R\ (a < b))$ (every element of L is less than all the elements of R)

4. $\forall x{\in}L\ (\exists y{\in}L\ (x < y))$ (L contains no greatest element)

5. $\forall x{\in}R\ (\exists y{\in}R\ (y < x))$ (R contains no smallest element)

If condition 5 is omitted, then the remaining four conditions define a *real number*. **Dedekind cuts, (*L*, *R*), representing only irrational numbers will be considered in this document**. The elements of partition L are all on the left of the cut, and the elements of partition R are all on the right of the cut on the number line of rational numbers.

In the diagram below, the rational numbers are represented by the area between the two horizontal lines. The rational numbers increase to the right.



Dedekind cut
(represents an irrational number)

The vertical bar represents a Dedekind cut — an irrational number. The L and R represent sets as defined in the definition of a Dedekind cut above. All the Dedekind cuts represent all the irrational numbers and vice versa. The two sets contain only rational numbers.

Side Note: The reader should be made aware at this point that doomsday is fast approaching for *transfinite cardinal numbers* — repent now, for that time is nigh.

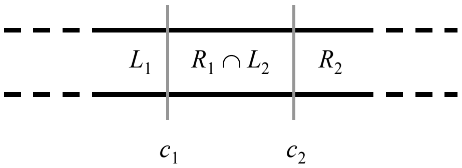
An important property of Dedekind cuts is that between any two irrational numbers, c_1 and c_2, there is a rational number.

Theorem 10: There exists a rational number between any two distinct Dedekind cuts (irrational numbers).

Proof:

Let $c_1 = (L_1, R_1)$ and $c_2 = (L_2, R_2)$. Assume that $c_1 < c_2$.

We seek to prove that the set of rational numbers between c_1 and c_2 is not the null set, therefore it contains at least one rational number. In other words, we seek to prove $R_1 \cap L_2 \neq \varnothing$.



$c_1 < c_2 \overset{\text{def}}{=} L_1 \subset L_2$. [by definition from the theory of Dedekind cuts]

$L_1 \subset L_2 \Rightarrow (\overline{L}_1 \cap L_2 \neq \varnothing)$. [from set theory]

$\overline{L}_1 \cap L_2 = R_1 \cap L_2$. [since L_1 and R_1 are complementary from the definition of a Dedekind cut]

$R_1 \cap L_2 \neq \varnothing$. [substitution]

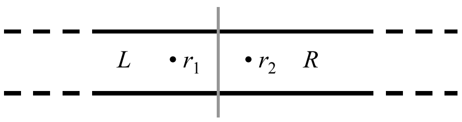
$\exists x\ (x \in (R_1 \cap L_2))$. [obviousness] ■

Another important property of Dedekind cuts is that between any two rational numbers, r_1 and r_2, there is an irrational number (a Dedekind cut).

Theorem 11: There exists an irrational number (Dedekind cut) between any two distinct rational numbers.

Proof:

Assume that $r_1 < r_2$. L is the set of rational numbers to the left of the Dedekind cut (the vertical line), and R is the set to the right.



$(r_2 - r_1)/\sqrt{2} + r_1$ is an irrational number because $\sqrt{2}$ is irrational and r_1 and r_2 are rational. Note that in place of the $\sqrt{2}$, any irrational number greater than 1 can be used.

Define the sets L and R: $L = \{x : x < (r_2 - r_1)/\sqrt{2} + r_1\}$ and $R = \{x : x > (r_2 - r_1)/\sqrt{2} + r_1\}$.

The Dedekind cut is the irrational number $(r_2 - r_1)/\sqrt{2} + r_1$ between r_1 and r_2. ■

Verification:

$(r_1 < (r_2 - r_1)/\sqrt{2} + r_1) \Leftrightarrow (0 < (r_2 - r_1)/\sqrt{2}) \Leftrightarrow (0 < (r_2 - r_1) \Leftrightarrow r_1 < r_2)$ □

$(r_2 > (r_2 - r_1)/\sqrt{2}) + r_1 \Leftrightarrow ((r_2 - r_1) > (r_2 - r_1)/\sqrt{2}) \Leftrightarrow (1 > 1/\sqrt{2}) \Leftrightarrow (\sqrt{2} > 1)$ □

The two theorems above prove that the rational and irrational numbers are interleaved, making it impossible that the two sets of numbers have different *cardinalities*.

2.2 The Irrationals Precede the Rationals

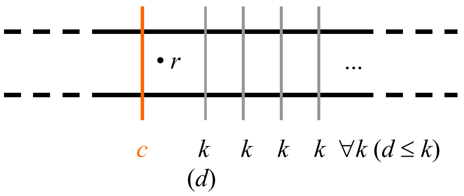
This section proves that the irrational numbers *precede* the rational numbers. This means that all of the irrational numbers can correspond one-to-one to a *subset* of the rational numbers. The set of rational numbers is denoted by $\mathbb{Q}$, and the set of irrational numbers is denoted by $\overline{\mathbb{Q}}$.

For any given irrational number, c, there is at least one rational number, r, greater than c and less than all other irrational numbers, k, greater than c. c can be associated uniquely with that rational number r. This is written mathematically as

$$\forall c{\in}\overline{\mathbb{Q}}\ (\exists r{\in}\mathbb{Q}\ (\forall k{\in}\overline{\mathbb{Q}}\ (c < k \rightarrow c < r < k))). \qquad (50)$$

Statement 50 implies $\overline{\mathbb{Q}} \precsim \mathbb{Q}$, therefore $|\overline{\mathbb{Q}}| \leq |\mathbb{Q}|$, because each c can be associated with one (or more) r uniquely among similar associations of all other irrational numbers k.

This is illustrated in the following diagram for each c. The area between the two horizontal lines represents the rational number line increasing to the right. The vertical lines represent Dedekind cuts between rational numbers. The dot represents the rational number(s) to the left of the cuts k.



All the cuts, k, to the right of the reference cut, c, have the one (or more) same rational number, r, to their left. This means that the reference cut c can be associated with a unique rational number (r) on its right, among all similar associations of all the cuts (irrational numbers) in the rational number line. This situation applies to each c in the number line. Therefore, each c in the number line can be associated with its own rational number, unique among the rational numbers associated with other cuts.

Justification of statement 50:

Let $c \in \overline{\mathbb{Q}}$. The statement $\forall d{\in}\overline{\mathbb{Q}}\ (\exists r{\in}\mathbb{Q}\ (c < d \rightarrow c < r < d))$ says that, if $c < d$, then between c and d, there is an r (for each d). This is proven by Theorem 10.

But, the same rational number r is also less than all irrational numbers, k, greater than or equal to the irrational number d (see diagram). In other words, if r is less than d, it is also less than all k's greater than or equal to d. The said statement is modified, as follows, to express this new fact.

$$\forall d{\in}\overline{\mathbb{Q}}\ (\exists r{\in}\mathbb{Q}\ (\forall k{\in}\overline{\mathbb{Q}}\ (c < d \leq k \rightarrow c < r < d \leq k))).$$

We are not interested in d, so we equate d with k and remove $\forall d{\in}\mathbb{Q}$. Therefore,

$$\exists r{\in}\mathbb{Q}\ (\forall k{\in}\overline{\mathbb{Q}}\ ((d = k) \wedge (c < d \leq k \rightarrow c < r < d \leq k))) \Rightarrow$$

$$\exists r{\in}\mathbb{Q}\ (\forall k{\in}\overline{\mathbb{Q}}\ (c < k \leq k \rightarrow c < r < k \leq k)) \Rightarrow \quad \text{[substitute } k \text{ for } d\text{]}$$

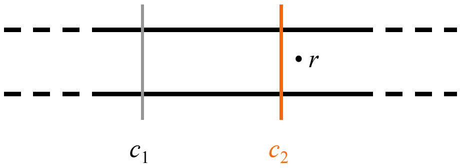
$$\exists r{\in}\mathbb{Q}\ (\forall k{\in}\overline{\mathbb{Q}}\ (c < k \rightarrow c < r < k)). \quad \text{[simplify]}$$

The above applies to all $c \in \overline{\mathbb{Q}}$. Therefore,

$$\forall c{\in}\overline{\mathbb{Q}}\ (\exists r{\in}\mathbb{Q}\ (\forall k{\in}\overline{\mathbb{Q}}\ (c < k \rightarrow c < r < k))). \blacksquare$$

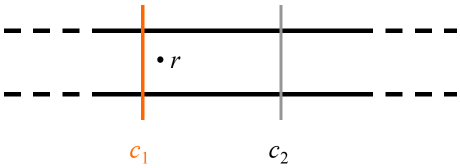
Statement 50 is justified, and each c can be associated with a unique r to its right.

Can two or more cuts (irrational numbers), c_1 and c_2, be associated with the same rational number r? Consider that $c_1 < c_2$, and $c_2 < r$. Assume that r is associated with c_2.



Now, r cannot be associated with c_1 because it is to the right of c_2; the rational number associated with c_1 must be to the left of <u>all</u> cuts to its right, and r is not to the left of c_2. In other words, for r to be associated with c_1 it must be between c_1 and c_2 ($c_1 < r < c_2$).

Now assume that r is associated with c_1.



We see that r cannot be associated with c_2 because the rational number associated with c_2 must be to the right of c_2.

Therefore, each cut (irrational number) corresponds to a unique rational number among the rational numbers associated with all other cuts (irrational numbers). A cut can possibly be associated with more than one rational number, but no other cut can be associated with those rational numbers. So, we have that each irrational number can correspond one-to-one with (at least) one rational number. Therefore, the irrational numbers *precede* the rational numbers. In other words, $\overline{\mathbb{Q}} \precsim \mathbb{Q}$, as promised.

2.3 The Rationals Precede the Irrationals

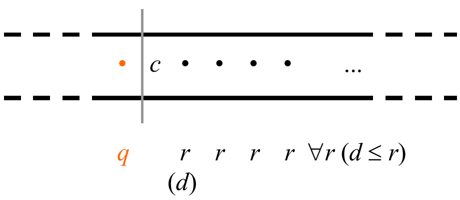
This section proves that the rational numbers *precede* the irrational numbers. This means that <u>all</u> of the rational numbers can correspond one-to-one to a *subset* of the irrational numbers. The set of rational numbers is denoted by $\mathbb{Q}$, and the set of irrational numbers is denoted by $\overline{\mathbb{Q}}$.

For any given rational number, q, there is at least one irrational number, c, greater than q and less than <u>all</u> other rational numbers, r, greater than q. q can be associated uniquely with that irrational number c. This is written mathematically as

$$\forall q \in \mathbb{Q}\ (\exists c \in \overline{\mathbb{Q}}\ (\forall r \in \mathbb{Q}\ (q < r \rightarrow q < c < r))) \qquad (51)$$

Statement 51 implies $\mathbb{Q} \precsim \overline{\mathbb{Q}}$, therefore, $|\mathbb{Q}| \le |\overline{\mathbb{Q}}|$, because each r can be associated with one (or more) c uniquely among similar associations of all other rational numbers r.

This is illustrated in the following diagram for each q. The area between the two horizontal lines represents rational numbers increasing to the right. The dots represent rational numbers. The vertical line represents the Dedekind cut(s) to the left of the rational numbers r.



<u>All</u> the rational numbers, r, to the right of the reference rational number, q, have one (or more) same cut (irrational number), c, to their left. This means that the reference rational number q can be associated with a unique cut (c) on its right, among similar associations of all the rational

numbers in the rational number line. This situation applies to each q in the number line. Therefore, each q in the number line can be associated with its own cut (irrational number), unique among the cuts associated with other rational numbers.

Justification of statement 51:

Let $q \in \mathbb{Q}$. The statement $\forall d{\in}\mathbb{Q}\ (\exists c{\in}\overline{\mathbb{Q}}\ (q < d \rightarrow q < c < d))$ says that, if $q < d$, then between q and d, there is a c (for each d). This is proven by Theorem 11.

But, the same irrational number c is also less than all rational numbers, r, greater than or equal to the rational number d (see diagram). In other words, if c is less than d, it is also less than all r's greater than or equal to d. The said statement is modified, as follows, to express this new fact.

$\forall d{\in}\mathbb{Q}\ (\exists c{\in}\overline{\mathbb{Q}}\ \ (\forall r{\in}\mathbb{Q}\ (q < d \leq r \rightarrow q < c < d \leq r)))$.

We are not interested in d, so we equate d with r and remove $\forall d{\in}\mathbb{Q}$. Therefore,

$\exists c{\in}\overline{\mathbb{Q}}\ \ (\forall r{\in}\mathbb{Q}\ ((d = r) \wedge (q < d \leq r \rightarrow q < c < d \leq r))) \Rightarrow$

$\exists c{\in}\overline{\mathbb{Q}}\ \ (\forall r{\in}\mathbb{Q}\ (q < r \leq r \rightarrow q < c < r \leq r)) \Rightarrow$ [substitute r for d]

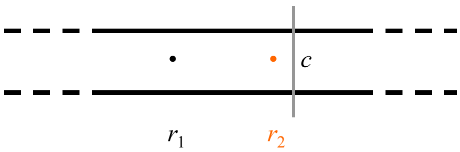
$\exists c{\in}\overline{\mathbb{Q}}\ \ (\forall r{\in}\mathbb{Q}\ (q < r \rightarrow q < c < r))$. [simplify]

The above applies to all $q \in \mathbb{Q}$. Therefore,

$\forall q{\in}\mathbb{Q}\ (\exists c{\in}\overline{\mathbb{Q}}\ \ (\forall r{\in}\mathbb{Q}\ (q < r \rightarrow q < c < r)))$. ■

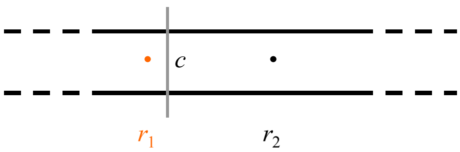
Statement 51 is justified, and each q can be associated with a unique c on its right.

Can two or more rational numbers, r_1 and r_2, be associated with the same irrational number c? Consider that $r_1 < r_2$, and $r_2 < c$. Assume that c is associated with r_2.



Now, c cannot be associated with r_1 because it is to the right of r_2; the irrational number associated with r_1 must be to the left of all rational numbers to its right, and c is not to the left of r_2. In other words, for c to be associated with r_1 it must be between r_1 and r_2 ($r_1 < c < r_2$).

Now assume that c is associated with r_1.



We see that c cannot be associated with r_2 because the irrational number associated with r_2 must be to the right of r_2.

Therefore, each rational number corresponds to a unique irrational number (cut) among the irrational numbers associated with all other rational numbers. A rational number can possibly be associated with more than one irrational number, but no other rational number can be associated with those irrational numbers. So, we have that each rational number can correspond one-to-one with (at least) one irrational number. Therefore, the rational numbers *precede* the irrational numbers. In other words, $\mathbb{Q} \precsim \overline{\mathbb{Q}}$, as promised.

2.4 Other Arguments

There are other intuitive arguments to show that the *cardinalities* of the rational numbers and irrational numbers are the same. These are not rigorous arguments, but nonetheless point in the direction that makes the justification of *transfinite cardinal numbers* beyond $\aleph_0$ doubtful.

Note that the preceding sections will later lead to the absolute certainty that the existence of *transfinite cardinal numbers* beyond $\aleph_0$ is factitious.

2.4.1 Cardinalities of Rational and Irrational Numbers

This argument uses the definition of an irrational Dedekind cut (see 2.1_Preliminaries).

That the *cardinality* of the set of irrational Dedekind cuts (irrational numbers) cannot exceed the *cardinality* of the set of rational numbers is proven as follows.

Each irrational Dedekind cut is associated with a unique set to its left ("L-set"). For any given *L-set* (S_L), consider all the *L-sets* to its left (S_1, S_2, S_3, …) on the rational number line. The given *L-set* minus all the *L-sets* to its left ($S_L \backslash (S_1 \cup S_2 \cup S_3 \cup \ldots)$) cannot be the null set (by definition of a Dedekind cut). That set difference (S_D) contains some rational numbers ($S_D \neq \varnothing$).

Now consider each possible *L-set* (S_L) on the rational number line. Each S_L corresponds to an irrational number and is also associated with a unique S_D because all the S_D's are disjoint (because of the definition of *set difference*). Therefore, the sets of *L-sets*, irrational numbers, and S_D's are *equivalent*.

Define a choice set (S_C) containing a single arbitrary member from each S_D (allowable by the Axiom of Choice). The *cardinality* of S_C cannot be greater than the *cardinality* of the rational numbers, because that choice set contains only rational numbers. But, the set of irrational numbers is *equivalent* to the set of irrational Dedekind cuts, which is *equivalent* to the set of all the S_D's, which is *equivalent* to the set S_C, which cannot be greater than the *cardinality* of the set of rational numbers, which is $\aleph_0$. Therefore, the *cardinality* of the set of irrational numbers cannot be greater than $\aleph_0$ — simples. In other words, the definition of irrational Dedekind cuts guarantees that the *cardinality* of the cuts (irrational numbers) cannot be greater than the *cardinality* of the rational numbers. That is to say, $|\overline{\mathbb{Q}}| \leq |\mathbb{Q}|$.

However, mathematicians claim that, at least, $|\mathbb{Q}| \leq |\overline{\mathbb{Q}}|$. So, in conclusion,

$(|\overline{\mathbb{Q}}| \leq |\mathbb{Q}|) \wedge (|\mathbb{Q}| \leq |\overline{\mathbb{Q}}|) \Rightarrow (|\mathbb{Q}| = |\overline{\mathbb{Q}}|)$.

2.4.2 Maximum Partition Size of the Set of Rational Numbers

The maximum *cardinality* of all possible partitions of a finite set cannot exceed the *cardinality* of the set. Each member of a partition is called a 'block'. We can therefore say that the maximum number of *blocks* in a finite set cannot exceed the *cardinality* of the set. For example, the maximum number of *blocks* in {5, 6, 7} cannot exceed 3; those blocks are {5}, {6}, and {7}; it is not possible to have a greater number of *blocks*. Likewise, for *infinite sets*, the maximum number of *blocks* cannot exceed the *cardinality* of the set because the maximum number of *blocks* are the singletons of the set elements.

The *cardinality* of the set of rational numbers is $\aleph_0$ ($|\mathbb{Q}| = \aleph_0$). Therefore, the maximum number of *blocks* of the set of rational numbers cannot exceed $\aleph_0$. However, based on the definition of irrational Dedekind cuts, each cut (irrational number) is between the *blocks* of the set of rational numbers, and for each *block* there can be only one unique cut associated with it (say, to its right). Therefore, the maximum number of cuts cannot exceed the maximum number of *blocks* which cannot exceed $\aleph_0$. In other words, the *cardinality* of the irrational numbers cannot exceed the *cardinality* of the rational numbers. That is to say, $|\overline{\mathbb{Q}}| \le |\mathbb{Q}|$.

Alternatively, if the *cardinality* of the set of irrational Dedekind cuts (irrational numbers) is greater than $|\mathbb{Q}|$ (as dogmatically asserted by mathematicians), then so is the *cardinality* of the set of *blocks* defined by those cuts greater, since each *block* is associated with exactly one cut. That contradicts the fact that the maximum number of *blocks* of the set of rational numbers cannot exceed $|\mathbb{Q}|$. Therefore, the *cardinality* of the set irrational numbers is <u>not</u> greater than $|\mathbb{Q}|$. That is to say, $|\overline{\mathbb{Q}}| \le |\mathbb{Q}|$.

However, mathematicians claim that, at least, $|\mathbb{Q}| \le |\overline{\mathbb{Q}}|$. So, in conclusion,

$(|\overline{\mathbb{Q}}| \le |\mathbb{Q}|) \wedge (|\mathbb{Q}| \le |\overline{\mathbb{Q}}|) \Rightarrow (|\mathbb{Q}| = |\overline{\mathbb{Q}}|)$.

2.4.3 Interleaving of Rational and Irrational Numbers

It is apparent from Theorems 10 and 11 that the rational and irrational numbers are interleaved. For any two distinct rational numbers, there is at least one irrational number in-between, and for any two distinct irrational numbers, there is at least one rational number in-between.

So, we have the set of *real numbers* as $\{\ldots, r_0, —, i_0, —, r_1, —, i_1, —, r_2, —, i_2, —, \ldots\}$, where the r_k's and i_k's are all different. Each r_k (a rational number) is associated with a unique i_k (an irrational number), regardless of how dense the r_k's and i_k's are. The interleaving ensures that the *cardinalities* of the two sets are identical. Therefore, $\aleph_0 = |\mathbb{Q}| = |\overline{\mathbb{Q}}|$. The only way that there could be more i_k's than r_k's is that if, for some pairs of irrational numbers, there is no rational number in-between. Because of the interleaving, there is no reason why the irrational numbers ought to "out number" the rational numbers (or vice versa).

In the limiting case of the interleaving, we have $\{\ldots, r_0, i_0, r_1, i_1, r_2, i_2, \ldots\}$ for each distinct rational and irrational number. Again, each rational number is associated with a unique irrational number, therefore, $\aleph_0 = |\mathbb{Q}| = |\overline{\mathbb{Q}}|$.

2.4.4 Axiom of Infinity Disallows Transfinite Cardinals

The Axiom of Infinity in ZFC set theory effectively defines the set of *natural numbers*, allowing the formation of *infinite sets*. That axiom does not allow the formation of sets that cannot be put to a one-to-one correspondence with it, such as the set of *real numbers*. It does not define sets that have *cardinalities* greater than $\aleph_0$. So, a set with a *cardinality* greater than the one implied by the Axiom of Infinity cannot be deduced from the ZFC axioms. To define such a set requires a new axiom of set theory. Any such set with higher *cardinality* will be inconsistent with the other axioms as shown in this document, so a new axiom is not an option.

So, in conclusion, $(|\mathbb{Q}| = |\overline{\mathbb{Q}}|)$.

Side Note: There is nothing in the definition of *equivalent* sets that prevents all *infinite sets* from being *equivalent* to each other. In other words, the definition of *equivalent* sets regards the elements of all *infinite sets* as indivisible units, and in that respect, they are indistinguishable from one another — the definition does not depend on the nature of the elements of the two sets. For example, the different natures of the rational numbers and the irrational numbers is irrelevant to the definition of *equivalence*. Consequently, two *infinite sets* always satisfy the definition of *equivalence*.

2.5 Cantor's Power Set Theorem Refuted

Georg Cantor [1845–1918], and his lackeys, believe in their hearts that the *cardinality* of any set is strictly less than the *cardinality* of its *power set* ($\forall A\ (|A| < |\wp(A)|)$). Such a belief is shamefully called "Cantor's (Power Set) Theorem". However, a set can be defined that is an exception to that theorem, making the "theorem" FALSE.

Theorem: There exists a set, A, such that $|A| = |\wp(A)|$

Proof:

$\exists A\ (\exists X\ (A = \wp(A) \cup X))$. [see below]

$\exists f\ (f \in \{h : h{:}A \to \wp(A)\} \wedge f(x) = \{x\})$. [since for each $x \in A$ there is $\{x\} \in \wp(A)$]

$\exists g\ (g \in \{h : h{:}\wp(A) \to A\} \wedge g(x) = x)$. [since $\wp(A) \subseteq A$ from the first step]

$A \precsim \wp(A)$. [since f is one-to-one and into]

$|A| \leq |\wp(A)|$. [from cardinality theory]

$\wp(A) \precsim A$. [since g is one-to-one and into]

$|\wp(A)| \leq |A|$. [from cardinality theory]

$|A| = |\wp(A)|$. [from the Schröder-Bernstein theorem] ■

The crucial step is the first one that states that the set A exists. Let $X = \{0\}$ to satisfy the existence of X in the first step of the proof. Then, the set A is defined as follows.

The set A satisfies the following conditions.

1. $0 \in A$.
2. $\varnothing \in A$.
3. $\forall x\ (x \in A \leftrightarrow \{x\} \in A)$
4. $\forall x_0, \ldots, x_n\ (x_0, \ldots, x_n \in A \leftrightarrow \{x_0, \ldots, x_n\} \in A)$ [for all $n \in \mathbb{N}^+$]
5. $\forall x_0, x_1, x_2, \ldots\ (x_0, x_1, x_2, \ldots \in A \leftrightarrow \{x_0, x_1, x_2, \ldots\} \in A)$

Conditions 2 to 5 show that all the members of $\wp(A)$ are also members of A. Condition 1 shows that the member of X is also a member of A. By definition of A, A contains only the members of $\wp(A)$ and X, therefore $A = \wp(A) \cup X$. The first step of the proof of the Theorem is therefore TRUE.

Even if the Theorem above is rejected, Cantors Theorem is still refuted by Theorem 12 below.

2.5.1 Power Sets And the Diagonal Method

Cantor's theorem can be refuted by using the diagonal method instead of the method shown in the preceding section. The following theorem says that, if a set has the same *cardinality* as that of the *natural numbers*, then the *cardinality* of the *power set* of that set is the same as the *cardinality* of the *natural numbers*. This is represented by the following theorem.

Theorem 12: $\forall S\ ((|A| = |\mathbb{N}|) \Rightarrow (|\wp(A)| = |\mathbb{N}|))$

Proof:

Each *subset* of A can be represented by a binary sequence where a `1` in the i^{th} position of the sequence indicates that the i^{th} element of A is in the *subset*, and a `0` indicates that it is not. A list of all the permutations of the binary sequence represents all the possible *subsets* of A. The *cardinality* of the list of such sequences represents the *cardinality* of the *power set* of A. The diagonal method can then be used on the list of sequences with the same failure (for the same reason) as when it is used with the *unit interval*.

As an illustration, for the set of *natural numbers*, {0, 1, 2, …}, we can theoretically form the following list of sequences (in no particular order).

0, 1, 2, ⋯ (elements of the set of *natural numbers*)

0, 0, 0, ⋯ (sequence represents no elements)

1, 0, 0, ⋯ (sequence represents the element 0)

0, 1, 0, ⋯ (sequence represents the element 1)

1, 1, 0, ⋯ (sequence represents the elements 0 and 1)

0, 0, 1, ⋯ (sequence represents the element 2)

1, 0, 1, ⋯ (sequence represents the elements 0 and 2)

0, 1, 1, ⋯ (sequence represents the elements 1 and 2)

1, 1, 1, ⋯ (sequence represents the elements 1, 2, and 3)

⋮ ⋮ ⋮ ⋮

It can be proven that the list of binary sequences is *equivalent* to the *unit interval*, therefore the *cardinality* of $\wp(A)$ is the same as that of $[0, 1]$. This document proves that the *cardinality* of $[0, 1]$ is the same as that of $\mathbb{N}$, contrary to the subjective belief and fantasy of virtually all mathematicians at the time of this writing. Note that each binary sequence corresponds to a number in base 2, which corresponds to a number in base 10 in the *unit interval*. ■

If $A = \mathbb{N}$, then by Theorem 12, $|\mathbb{N}| = |\wp(\mathbb{N})|$, thus disproving Cantor's theorem, $\forall A\ (|A| < |\wp(A)|)$.

2.6 The Death of the Transfinite Cardinals

It was proven in Chapter 1 that Cantor's diagonal argument for claiming that the *unit interval* is *non-denumerable* is invalid. It was proven in the first few sections of Chapter 2 that the set of irrational numbers *precede* the set of rational numbers ($\overline{\mathbb{Q}} \precsim \mathbb{Q}$), and vice versa ($\mathbb{Q} \precsim \overline{\mathbb{Q}}$). It was also proven in the same chapter that Cantor's theorem is FALSE ($\neg\forall A\ (|A| < |\wp(A)|)$). We now put these facts together to rid mathematics of the factitious idea of *transfinite cardinal numbers* (except for $\aleph_0$).

The Schröder-Bernstein Theorem, $(A \precsim B) \wedge (B \precsim C) \Rightarrow (A \sim B)$, proves that $(\overline{\mathbb{Q}} \precsim \mathbb{Q}) \wedge (\mathbb{Q} \precsim \overline{\mathbb{Q}}) \Rightarrow (\overline{\mathbb{Q}} \sim \mathbb{Q})$. And, by definition, $|\mathbb{Q}| = |\overline{\mathbb{Q}}|$. But mathematicians accept that $|\mathbb{Q}| = \aleph_0$. Therefore, $|\overline{\mathbb{Q}}| = \aleph_0$. That is to say that the irrational numbers are *denumerable*, contrary to the accepted belief by current mathematicians.

By definition, $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$, and so $|\mathbb{R}| = |\mathbb{Q} \cup \overline{\mathbb{Q}}| = |\mathbb{Q}| + |\overline{\mathbb{Q}}| = \aleph_0 + \aleph_0 = \aleph_0$, since $\mathbb{Q} \cap \overline{\mathbb{Q}} = \varnothing$. But, because $|\mathbb{N}| = |\mathbb{R}| = \aleph_0$, then $\mathbb{N} \sim \mathbb{R}$. So, **THE REAL NUMBERS ARE DENUMERABLE**. Who would have thought? — well, the mathematicians before Cantor would have thought.

Note that, so far, there was no mention of $\aleph_1$, $\aleph_2$, $\aleph_3$, and so forth. Those fictions were invented by Cantor's theorem (you know, the FALSE one). But, by a recursive application of Theorem 12, $|\wp(\ldots(\wp(A))\ldots)| = |\mathbb{N}| = |\mathbb{R}| = \aleph_0$ for all A *equivalent* to $\mathbb{N}$. Bye-bye, $\aleph_1$, $\aleph_2$, $\aleph_3$, and so forth.

3

Epilogue

The previous two chapters mark the death of *transfinite cardinal numbers* (except for $\aleph_0$) because all *infinite sets* are *denumerable*. The definition of *transfinite cardinal numbers* depended upon Cantor's theorem and the non-*denumerability* of the *real numbers*. This document has dismissed those two fantasies. There seems to be no hope of any other set being proven *non-denumerable*.

This now raises the question: How is it that mathematicians (and logicians and others) failed to recognise that the diagonal argument and the power set argument are invalid? And, how is it that they failed to recognise that the very definition of Dedekind cuts for irrational numbers proves that the set of irrational numbers is *equivalent* to the set of rational numbers?

The answers to these questions are speculative, but here are some possibilities.

1. What mathematicians may have done was to interpret the diagonal method independently of the whole argument (for example, they defined the method to construct a *real number* in the *unit interval* using an <u>arbitrary</u> digit at each decimal place), then attempted to project that method to the context of the antithesis without realising that the diagonal number was being used in its own definition in that context.
2. Perhaps they assumed, hypothetically, that the numbers in the *unit interval* are *denumerable*, then tried to "construct" a number (which they dogmatically asserted to exist) from that set, and then concluded that a number can always be "constructed" that does not belong to the set, so they concluded that such a *denumerable* set cannot exist. Here, again, they failed to realise that the diagonal number <u>would</u>, by the antithesis itself, be in that set. Therefore, no such number can be constructed (as shown in this document). Mathematicians dogmatically claim that the diagonal number is guaranteed to not be in the list of the antithesis, without realising that the construction method is self-contradictory. So, the only thing that the construction method guarantees is that the method is self-contradictory under the assumption of the antithesis. Similarly for the power set argument.
3. Perhaps, by presenting the list representing the antithesis, and assuming that the diagonal number exists but is not in the list, they failed to realise that the list is now a different list from the original antithesis list (by the implication of the diagonal method), but continued to assume that it is the same list because they fixated, in their minds, that it is the same list. As a consequence, they wrongly imagined that it was impossible for "the (original) list" to contain all the *reals* in the *unit interval*, when, in logical reality, it is only the <u>new</u> list that does not contain the diagonal number. In other words, they wrongly thought that the diagonal method applied to the original list, when, in fact, for the method to be valid, it applies to a *proper subset* of the original list but not the original list itself. Similarly for the power set argument.
4. Perhaps, mathematicians and logicians and others perceive some sort of mystique in the idea of having different "levels" of infinity, and refuse to admit that their delusion is just a fantasy. As a consequence, they stubbornly refuse to admit that there is anything wrong with the diagonal and power set arguments.
5. Because of the wording of the argument, using such words as "construct", "choose", and other misleading terms, they imagined that the diagonal argument is literally about trying to construct some sort of metaphysical or hypothetical list, and so they imagined from the presentation that such a list could not be constructed.
6. Perhaps, mathematicians and logicians are not as clever as they pretend to be, and do not quite understand what genuine logic is about (was it Bertrand Russell who allegedly said "Mathematics … is a subject in which we do not know what we are talking about …"?).

With the Dedekind cuts, maybe no one thought of using the irrational number version of the definition, or no one made the connection between the concept of *equivalence* and Dedekind cuts (admittedly, there is not an extremely obvious connection).

It is incredible that mathematicians and others cannot see an obvious flaw in a children's argument (the diagonal argument especially). The reason for that failure may be that mathematicians and logicians use too much unsubstantiated and confused subjective intuitions instead of logical reasoning. This is what happens when they give priority to their intuitions over genuine logical thinking.

Side Note: I find it personally disturbing to imagine how clueless I would have to be to not recognise the obvious self-contradiction in the diagonal method.

3.1 Consequences of the Death of the Transfinite Cardinals

Of course, there are some advantages and disadvantages with the death of the higher *transfinite cardinal numbers*.

3.1.1 Advantages

Some of the advantages of now realising that all *infinite sets* are *denumerable* include:

1. Mathematics will become more advanced as a result of the fact that there are no higher *transfinite cardinal numbers*, because it is always of great benefit to mathematics (and everything else) when bullshit is removed from it.
2. *Transfinite cardinal* arithmetic will now be simpler: if at least one of α and β are *transfinite cardinal numbers* then $\alpha + \beta = \alpha\beta = \alpha^\beta = \aleph_0$.
3. Certain so-called paradoxes that resulted from Cantor's power set "theorem" can now be ditched.
4. Mathematics lecturers can now shorten their courses on *transfinite cardinals*, and use the spare time to go fishing (or better still, to learn to use some rigorous mathematical logic).
5. Mathematics teachers now need not embarrass themselves trying to defend the indefensible when explaining the diagonal argument to intelligent students.
6. Young mathematical students now have something meaningful to write home about.

3.1.2 Disadvantages

Some of the disadvantages of now realising that all *infinite sets* are *denumerable* include:

1. Mathematics books need to be rewritten.
2. Mathematics courses need to be redesigned.
3. AI (artificial intelligence) needs to be retrained.
4. Mathematics dictionaries need to be rewritten.
5. Encyclopaedias need to be rewritten.
6. Internet websites need to be rewritten.
7. Apologies need to be made.

All this due to mathematical carelessness and incompetence in providing valid proofs.

Other disadvantages are:

1. Most mathematical theorems now need to be questioned given that most mathematicians cannot even see an obvious flaw in a children's (diagonal) argument.
2. The volumes of mathematical and philosophical books written on *transfinite cardinals* will need to be burned, adding to the problem with global climate change.
3. Psychiatrists will now become overworked healing all the bruised mathematical egos.

3.2 Concluding Remarks

So there we have it. The higher *transfinite cardinal numbers* — it was all just fantasy. So much for the peer review system (maybe it ought to be called the "peer lackey system"). Looks like the continuum has just lost its "power" (the one that it didn't have). And that *continuum hypothesis* thing — that nonsense can now be ditched.

In summary, both the diagonal and power set the arguments use a similar strategy, as follows. Both arguments attempt to prove a non-*equivalence* between two sets by utilising a proof by contradiction. The antithesis of the non-*equivalence* (ie: the *equivalence* of the two sets) involves a function. The arguments attempt to prove that the function cannot be a *bijection*, meaning that the function's range must be a *proper subset* of the function's co-domain (if the function is a *bijection*, its range and co-domain will be equal). The arguments attempt to "construct" an entity that exists in the co-domain but not in the range of the function. The contradiction resulting from the "construction" of that entity is claimed to be the contradiction required by the proof by contradiction, thereby concluding that function cannot be a *bijection*.

The flaw in both arguments is mainly this (as proven in this document):

1. The definition of the "constructed" entity is self-contradictory when used on the original (hypothesised) function's range (the range is assumed to be equal to the co-domain), so the entity cannot exist and the argument fails. Note that it is invalid to use the entity's self-contradiction as the contradiction required by the proof by contradiction.
2. The definition of the "constructed" entity is not self-contradictory when used only on a *proper subset* of the original (hypothesised) function's range. But, in that case, no contradiction with the original range arises, so the argument fails. That the entity is not a member of a *proper subset* of the range does not prove that the function's range cannot be equal to its co-domain (ie: that it cannot be a *bijection*).

Besides the invalid arguments, this document proves that all *infinite sets* are indeed *equivalent* anyway.

So, how long will it take mathematicians (and others) to cast *transfinite cardinals* into the bowel movements of history (where they belong)? — 50 years? 100 years? 200 years? Such is the stubbornness of human ignorance.

The Final Word: We can say that the *transfinite cardinals* bled to death from Dedekind cuts (except for the sole survivor, $\aleph_0$, poor thing, all alone in this big wide world of mathematics).

ℵℵℵ TRANSFINITE CARDINALS ℵℵℵ

☺ **Rest In Peace** ☺

may the power of the continuum abandon them

Appendix A

Proof of Theorems

This appendix presents the proofs of Theorems 1, 2, and 3.

A.1 Proof of Theorem 1

Proof of Theorem 1: $\exists r{\in}S\ (\forall x{\in}S\ (r \neq x)) \Leftrightarrow \bot$

Proof 1:

$\exists r{\in}S\ (\forall x{\in}S\ (r \neq x)) \Leftrightarrow$ [left expression of Theorem]

$\exists r{\in}S\ (\forall x{\in}\{r\}\ (r \neq x) \wedge \forall x{\in}S\backslash\{r\}\ (r \neq x)) \Leftrightarrow$ [separation]

$\exists r{\in}S\ ((r \neq r) \wedge \forall x{\in}S\backslash\{r\}\ (r \neq x)) \Leftrightarrow$ [simplification]

$\exists r{\in}S\ (\bot) \Leftrightarrow \bot$. [because $(r \neq r) \Leftrightarrow \bot$]

$\exists r{\in}S\ (\forall x{\in}S\ (r \neq x)) \Leftrightarrow \bot$ [deduction from steps above] ■

Proof 2:

$(\forall r{\in}S\ (\exists x{\in}S\ (r = x)) \Leftrightarrow \top) \Rightarrow$ [tautology]

$(\neg\forall r{\in}S\ (\exists x{\in}S\ (r = x)) \Leftrightarrow \neg\top) \Rightarrow$ [negation]

$(\exists r{\in}S\ (\forall x{\in}S\ (r \neq x)) \Leftrightarrow \bot)$ [De Morgan] ■

Proof 3:

Lemma: $a \in A \Leftrightarrow \exists x{\in}A\ (a = x)$. [tautology]

$(\forall r{\in}S\ (r \in S) \Leftrightarrow \top)$ [tautology]

$(\forall r{\in}S\ (\exists x{\in}S\ (r = x)) \Leftrightarrow \top)$ [by Lemma]

$(\neg(\neg\forall r{\in}S\ (\exists x{\in}S\ (r = x))) \Leftrightarrow \top)$ [double negation]

$(\neg\exists r{\in}S\ (\forall x{\in}S\ (r \neq x)) \Leftrightarrow \top)$ [De Morgan]

$(\exists r{\in}S\ (\forall x{\in}S\ (r \neq x)) \Leftrightarrow \bot)$ [negation] ■

A.2 Proof of Theorem 2

Proof of Theorem 2: $(S_2 = S_1) \Rightarrow \neg\exists r{\in}S_1\ (\forall x{\in}S_2\ (r \neq x))$

Lemma: $a \in A \Leftrightarrow \exists x{\in}A\ (a = x)$ [tautology]

$S_2 = S_1 \Rightarrow$ [antecedent of Theorem]

$\forall y{\in}S_2\ (y \in S_1) \wedge \forall r{\in}S_1\ (r \in S_2) \Rightarrow$ [definition of antecedent]

$\forall r \in S_1\ (r \in S_2) \Rightarrow$ [deduction]

$\forall r \in S_1\ (\exists x \in S_2\ (r = x)) \Rightarrow$ [by Lemma]

$\neg(\neg \forall r \in S_1\ (\exists x \in S_2\ (r = x))) \Rightarrow$ [double negation]

$\neg \exists r \in S_1\ (\forall x \in S_2\ (r \neq x))$. [De Morgan]

$(S_2 = S_1) \Rightarrow \neg \exists r \in S_1\ (\forall x \in S_2\ (r \neq x))$ [deduction from steps above] ■

A.3 Proof of Theorem 3

Proof of Theorem 3: $(S_2 \subset S_1) \Rightarrow \exists r \in S_1\ (\forall x \in S_2\ (r \neq x))$

Lemma 1: $a \in A \Leftrightarrow \exists x \in A\ (a = x)$ [tautology]

Lemma 2: $a \notin A \Leftrightarrow \forall x \in A\ (a \neq x)$ [by negation of both sides of Lemma 1]

$S_2 \subset S_1 \Rightarrow$ [antecedent]

$S_2 \subseteq S_1 \wedge S_2 \neq S_1 \Rightarrow$ [definition of antecedent]

$\forall z \in S_2\ (z \in S_1) \wedge \neg(\forall y \in S_2\ (y \in S_1) \wedge \forall r \in S_1\ (r \in S_2)) \Rightarrow$ [definition expansion]

$\forall z \in S_2\ (z \in S_1) \wedge (\neg \forall y \in S_2\ (y \in S_1) \vee \neg \forall r \in S_1\ (r \in S_2)) \Rightarrow$ [De Morgan]

$\forall z \in S_2\ (z \in S_1) \wedge \neg \forall y \in S_2\ (y \in S_1) \vee \forall z \in S_2\ (z \in S_1) \wedge \neg \forall r \in S_1\ (r \in S_2) \Rightarrow$ [distribution]

$\perp \vee \forall z \in S_2\ (z \in S_1) \wedge \exists r \in S_1\ (r \notin S_2) \Rightarrow$ [complement and De Morgan]

$\exists r \in S_1\ (r \notin S_2) \Rightarrow$ [deduction]

$\exists r \in S_1\ (\forall x \in S_2\ (r \neq x))$. [by Lemma 2]

$(S_2 \subset S_1) \Rightarrow \exists r \in S_1\ (\forall x \in S_2\ (r \neq x))$ [deduction from steps above] ■

Side Note: If you're a pure mathematician and don't understand the mathematical logic above, I suggest that you get yourself educated in formal mathematical logic before you pretend to be an expert in understanding and producing mathematical proofs (especially Cantor's diagonal and power set arguments, and similar arguments).

Appendix B

Proof by Contradiction

This appendix shows the proper meaning of a proof by contradiction, and the proper process to conduct such a proof. Unfortunately, the meaning and logic underlying a proof by contradiction seems to be misunderstood by most mathematicians (and logicians). This appendix explains the proper logical basis and strategy for such a proof.

B.1 Introduction

A proof in pure mathematics is basically a sequence of statements that are logically derived from any of the axioms (typically ZFC axioms), possibly in conjunction with other conditions, concluding in a final statement. The final statement is called a "theorem", which may itself be a deductive implication. The most important point here is that all the statements must be logically derived, meaning that they must be deductive implications. For example, if A represents any of the axioms and T represents a theorem, then the proof of the theorem would be represented as ‹$A \Rightarrow T$›. Alternatively, if A represents any of the axioms, C represents a condition, and F represents the final statement, then the proof of the theorem ‹$C \Rightarrow F$› would be represented as ‹$A \Rightarrow (C \Rightarrow F)$›. Note that ‹$A \Rightarrow (C \Rightarrow F)$› is logically equivalent to ‹$(A \wedge C) \Rightarrow F$›. Of course, a proof may involve a large number of steps before the theorem is finally deduced. It is absolutely important to note that **NO arbitrary assumptions** are allowed in a proof.

A deductive implication is an implication that is logically necessarily TRUE. "Necessarily TRUE" means TRUE for all possible logic values (commonly called "truth values") of the involved statements. For example, ‹$(A \wedge B) \Rightarrow A$› is a deductive implication because the statement is TRUE for all logic values of A and B. The statement ‹$(A \wedge B) \Rightarrow C$› is not a deductive implication because it is not TRUE for all logical values of A, B, and C; it is correctly represented as ‹$(A \wedge B) \nRightarrow C$›.

The authoritative definition of a deductive implication is

$A \Rightarrow B \ =_{\text{df}} ((A \wedge \neg B) \Leftrightarrow \bot)$,

where $P \Leftrightarrow Q \ =_{\text{df}} \Box(P \wedge Q \vee \neg P \wedge \neg Q)$, and $\Box$ means 'the argument is necessarily TRUE'.

For completeness, the definition of a contingent implication is

$A \rightarrow B \ =_{\text{df}} ((A \wedge \neg B) \leftrightarrow \bot)$,

where $P \leftrightarrow Q \ =_{\text{df}} (P \wedge Q \vee \neg P \wedge \neg Q)$.

The following two definitions may also be useful.

$\neg(\Box A) \ =_{\text{df}} \Diamond(\neg A)$ and $\Box(\neg A) \ =_{\text{df}} \neg(\Diamond A)$,

where $\Diamond$ means 'not FALSE for all possible logic values of the argument'.

Side Note: Note that there is no talk of "possible worlds" here — such a fanciful notion is completely unnecessary and not required by the laws of logic.

Fun Fact: $(\bot \wedge \neg A) \Leftrightarrow (A \wedge \bot) \Leftrightarrow \bot$. So, ‹$\bot \Rightarrow A$› and ‹$A \Rightarrow \top$› by definition, for any A. Therefore, $(A \Rightarrow \bot) \Leftrightarrow (A \Leftrightarrow \bot)$.

There are a few techniques for proving a theorem in logic and pure mathematics. One of those techniques is called "proof by contradiction". This technique directly utilises the actual definition of a deductive implication even though it is traditionally called an "indirect proof".

B.2 The Meaning of a Proof by Contradiction

A proof by contradiction is simply the application of the definition of a deductive implication. For example, to prove $\langle(C \wedge D) \Rightarrow C\rangle$, the definition of deductive implication is applied to get $\langle(C \wedge D \wedge \neg C) \Leftrightarrow \perp\rangle$, which, by inspection, is a TRUE statement. If A represents any of the axioms, and T represents a theorem, then to prove the theorem from the axioms ($A \Rightarrow T$), all that is required is to prove $\langle(A \wedge \neg T) \Leftrightarrow \perp\rangle$. So, if the conjunction of the negation of a theorem and the axioms deductively imply a necessary falsehood (typically miscalled a "contradiction"), then, by definition, the axioms deductively imply the theorem. Of course, proof by contradiction can be used for arbitrary deductive implications. For example, to prove $\langle P \Rightarrow Q\rangle$ by contradiction requires the proof of $\langle(P \wedge \neg Q) \Leftrightarrow \perp\rangle$ for any statements P and Q. In Classical Logic and Mathematics this is expressed as "the conjunction of P and $\neg Q$ is a contradiction".

Note especially that there was no mention of "assuming the contrary" — no assumptions whatsoever have been utilised in the paragraph above. A proof by contradiction is purely a deductive process by applying the actual definition of a deductive implication; there are no assumptions to be made, hypothetically or otherwise.

B.3 The Proper Procedure for the Proof

Consider the deductive implication, $A \Rightarrow T$, to be proven by contradiction. T will be called the 'thesis', and its negation, $\neg T$, will be called the 'antithesis'. The intention is to determine whether $\langle(A \wedge \neg T) \Leftrightarrow \perp\rangle$. Typically, it is not immediately obvious whether $\langle A \wedge \neg T\rangle$ is necessarily FALSE (ie: a contradiction). In practice, some deductions need to be made from the conjunction so that it does become obvious that it is necessarily FALSE.

Now a potential problem can arise in the deductions. An inept mathematician (or logician) may introduce an arbitrary contradiction into those deductions (as was done with the diagonal and power set arguments), thereby feigning that the proof has succeeded. As a safeguard, the following process should be adopted.

Two distinct lines of derivations need to be made, one from the axioms (A) and another from the *antithesis* ($\neg T$). Neither line of logic is to involve the *thesis*, and the axiom line <u>must not</u> involve the *antithesis* directly or indirectly (if the *antithesis* is removed, the line of logic should still be valid). The conclusions from both lines need to be contradictory together, not separately.

Symbolically, we have the line from the axioms, $A \Rightarrow \ldots \Rightarrow C_1$, and the line from the *antithesis*, $\neg T \Rightarrow \ldots \Rightarrow C_2$. C_1 and C_2 would be simple enough to conclude that $\langle(C_1 \wedge C_2) \Leftrightarrow \perp\rangle$ (ie: that C_1 and C_2 are contradictory) by inspection. (It is known that if $\langle(C_1 \wedge C_2) \Leftrightarrow \perp\rangle$ then $\langle(A \wedge \neg T) \Leftrightarrow \perp\rangle$.) If the contradiction is proven, then $\langle A \Rightarrow T\rangle$ will have been proven as well (by definition).

If a deductive implication of the form $\langle A \Rightarrow (C \Rightarrow T)\rangle$ is to be proven by contradiction, then the logically equivalent deduction, $\langle(A \wedge C) \Rightarrow T)\rangle$, can be used. The axiom line would then be deductions from $(A \wedge C)$, and the *antithesis* would be $\neg T$.

To ensure that the contradiction is between the axioms and the *antithesis*, it is ABSOLUTELY important to ensure that each line of logic is logically valid. This means that no assumptions are to be made; each statement in each line of logic must be logically derived and validated. The *antithesis* is not assumed to be TRUE; it is simply considered, as required by the definition of deductive implication. It is also ABSOLUTELY important that neither line of logic results in a necessary falsehood (contradiction). If either line of logic results in a necessary falsehood, then, of course, that would imply ‹$(C_1 \wedge C_2) \Leftrightarrow \bot$›. But, in that case, the contradiction would be the result of an introduced contradiction, rather than the result of A and $\neg T$ being contradictory together. If a necessary falsehood does result from one of the lines of logic, then either an error has been made or the lines are not separated enough.

Both the diagonal and power set arguments are examples of an invalid proof by contradiction. In both cases, the contradiction is derived from the *antithesis* line (because it involves the *antithesis*), not from a conjunction of the axioms with the *antithesis*. Furthermore, in both cases, an arbitrary assumption is made by dogmatically assuming the existence of the "constructed" entity (the definition of the entity is defined to be self-contradictory under the condition of the *antithesis*, so no such entity can exist). In short, the contradiction is introduced (intentionally?), not deduced.

Another way to reduce the possibility of error is to use mathematical logical rather than clever intuitive wordings. For Classical Logic and Mathematics, this rules out "constructive" proofs. Such proofs are subjective and can be misleading (as we have seen in the diagonal and power set arguments); construction is best left for the building industry. A privately conceived "constructive" proof should finally be converted to a proper logical proof using mathematical logic or equivalent wording before being presented to the public.

A simple illustration of using proof by contradiction the proper way follows.

Theorem: $(a < b) \wedge (b < c) \Rightarrow a < c$.

Proof by contradiction.

Antithesis: $a \not< c$.

From the antithesis: $(a \not< c) \Rightarrow (a \geq c)$.

From the axioms: $(a < b) \Rightarrow (b - a > 0)$. $(b < c) \Rightarrow (c - b > 0)$. Therefore $(b - a) + (c - b) > 0$, since the addition of two positive numbers is a positive number. But, $(b - a) + (c - b) = c - a$. Therefore $c - a > 0$. But $(c - a > 0) \Rightarrow (c > a) \Rightarrow (a \not\geq c)$.

The antithesis contradicts the axioms, therefore $(a < b) \wedge (b < c) \Rightarrow a < c$.

Notice, firstly, that no assumption of anything being TRUE has been made in the proof. Secondly, there are two distinct lines of logic; one from the *antithesis*, and another from the axioms (and the antecedent of the theorem). If either line of logic had resulted in a contradiction, then an error would have been made somewhere in the argument (sound familiar with the diagonal and power set arguments?). Alternatively, the two lines of logic were not properly separated. Thirdly, and most importantly, the line of logic from the axioms (and antecedent) does not involve the *antithesis* in any way.

B.4 The Traditional Procedure for the Proof

In Classical Logic and Mathematics, an 'assumption' is a statement that is taken to be TRUE for the purpose of an argument, and typically used as the antecedent in an implication. An assumption can also be made when a statement is split into possible cases, one of which is the TRUE case.

However, the expression "A is TRUE" is ambiguous. It could mean 'A = TRUE', or '$A \leftrightarrow \top$'. The former means that A indicates only the value TRUE. The latter is equivalent to just A ($(A \leftrightarrow \top) \Leftrightarrow A$). Conversely, "$A$ is FALSE" could mean 'A = FALSE', or '$A \leftrightarrow \bot$' (or just $\neg A$).

In a traditional proof by contradiction, the negation of the conclusion (*antithesis*) of a theorem is "assumed to be (hypothetically) TRUE" for the purpose of the proof. This business of assuming that the *antithesis*, $\neg T$, is TRUE can only mean $\langle \neg T \leftrightarrow \top \rangle$, that is, it just means that $\neg T$ is being considered. It cannot mean $\langle \neg T = \text{TRUE} \rangle$ because that would be to assume that $\neg T$ has only one logic value (TRUE), which would eventually be proven to be the wrong value. However, mathematicians do seem to mean $\langle \neg T = \text{TRUE} \rangle$, and then give some sort of convoluted explanation of why such an assumption implies its exact opposite. So, assuming that the *antithesis* is TRUE is just verbiage that has no logical merit in itself, but is just an indication that the consideration of the *antithesis* is a requirement by the proof by contradiction. For technical accuracy, in a traditional proof by contradiction, one should merely "consider" the *antithesis*, not "assume that the *antithesis* is TRUE". In any case, in a logical deduction, there should not be any assumptions of TRUE or FALSE, other than when considering separate cases of logic values.

The most serious drawback in the traditional proof by contradiction is the absence of distinctly and clearly separating the *antithesis* lines of logic from the non-*antithesis* lines. The problem here is that inept mathematicians could inadvertently (and even deliberately) introduce a self-contradictory definition, claiming that the resulting contradiction is the one required by the proof by contradiction (there is no need to give two examples here!).

In summary, in a traditional proof by contradiction, **(1)** do not use meaningless expressions like "assume such-and-such [the *antithesis*] is TRUE" (say "consider such-and-such [the *antithesis*]" instead), **(2)** distinctly separate the lines of logic that do <u>not</u> involve the *antithesis* clearly from the lines of logic that do, making sure that no contradiction results from either of those two sets of lines. In short, use the proper method of proof by contradiction as described in the previous section.

Bibliography

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Glossary

B

bijection

A function, f, such that $f{:}A \to B$ is one-to-one and onto. Consequently, there is a one-to-one correspondence between A and B, and also A and B are *equivalent* ($A \sim B$).

C

cardinal number $|A|$

The family of sets that are *equivalent* to a given set, A, denoted by $|A|$. There are other equivalent ways of defining *cardinal number*. The *cardinal number* of the set of *natural numbers*, $\mathbb{N}$, is denoted by $\aleph_0$ (ie: $|\mathbb{N}| = \aleph_0$).

$|A| =_{\mathrm{df}} \{X : X \sim A\}$

cardinality

The *cardinal number* of a set.

D

denumerable

An *infinite set* that is *equivalent* to the set of *natural numbers*, $\mathbb{N}$. An *infinite set*, A, is said to be *denumerable* if $A \sim \mathbb{N}$.

E

equivalent (sets) $A \sim B$

Two sets, A and B, are said to be *equivalent* if it is possible to put them into a one-to-one correspondence with each other, denoted by $A \sim B$. The statement $A \nsim B$ denotes that two sets, A and B, are not *equivalent* (it is impossible to put them into a one-to-one correspondence with each other). Note: two *equivalent* sets have the same *cardinality*.

$A \sim B =_{\mathrm{df}}$

$\exists f\,(f \in \{g : g{:}A \to B\} \wedge \forall y{\in}B\,(\exists x{\in}A\,(f(x) = y)) \wedge \forall x_1, x_2{\in}A\,((f(x_1) = f(x_2)) \to (x_1 = x_2)))$

F

finite cardinal numbers

The set of *cardinal numbers* that are less than $\aleph_0$, $\{x : x < \aleph_0\}$. These are the *cardinal numbers* for finite sets.

I

infinite set

The set A is said to be *infinite* if there exists X such that $X \subset A$ and $X \sim A$.

N

natural numbers $\mathbb{N}$

The set of positive whole numbers together with the number zero, denoted by $\mathbb{N}$, or without the number zero, denoted by $\mathbb{N}^+$.

$\mathbb{N} =_{\text{df}} \{0, 1, 2, \ldots\} \qquad \mathbb{N}^+ =_{\text{df}} \{1, 2, 3, \ldots\}$

non-denumerable

An *infinite set* that is not *equivalent* to $\mathbb{N}$. An *infinite set*, A, is said to be *non-denumerable* if $A \nsim \mathbb{N}$.

P

power set $\wp(A)$

The set of all *subsets* of a set, A, denoted by $\wp(A)$.

$\wp(A) =_{\text{df}} \{X : X \subseteq A\}$

precede (sets) $A \precsim B$

A set A is said to *precede* a set B if there exists X such that $X \subseteq B$ and $X \sim A$. $A \precsim B$ denotes that A *precedes* B. If $A \precsim B$ then, by definition, $|A| \leq |B|$.

proper subset $A \subset B$

The set, A, whose entire members are also members of a set, B, where A is not equal to B, denoted by $A \subset B$.

$A \subset B =_{\text{df}} \forall x\, (x \in A \rightarrow x \in B) \wedge (A \neq B)$

R

real numbers $\mathbb{R}$

The union of the set of rational and irrational numbers, denoted by $\mathbb{R}$.

S

strictly precede (sets) $A \prec B$

A set A is said to *strictly precede* a set B if A *precedes* B and A is not *equivalent* to B ($A \precsim B \wedge A \nsim B$). $A \prec B$ denotes that A *strictly precedes* B. If $A \prec B$ then, by definition, $|A| < |B|$.

subset $A \subseteq B$

The set, A, whose entire members are also members of a set, B, denoted by $A \subseteq B$.

$A \subseteq B =_{\text{df}} \forall x\, (x \in A \rightarrow x \in B)$

T

transfinite cardinal number

The set of *cardinal numbers* that are greater than or equal to $\aleph_0$, $\{x : \aleph_0 \leq x\}$. These are the *cardinal numbers* for *infinite sets*.

U

unit interval

The *real numbers* between zero and one, inclusively, denoted by the closed interval $\mathbf{[}0, 1\mathbf{]}$. Note that the brackets are in bold type when indicating open or closed intervals.

$\mathbf{[}0, 1\mathbf{]} =_{\mathrm{df}} \{x : 0 \leq x \wedge x \leq 1\}$

Mathematical Symbols

$\lvert A \rvert$	The *cardinal number* of A.
$A \wedge B$	Logical *conjunction* of A and B.
$A \vee B$	Logical *disjunction* of A and B.
$\neg A$	Logical *negation* of A.
$\top$	*Necessary truth*.
$\bot$	*Necessary falsehood*.
$A \Rightarrow B$	*Logical implication* between A and B. Also called a *deductive implication*.
$A \Leftrightarrow B$	*Logical equivalence* between A and B.
$A \rightarrow B$	*Contingent implication* between A and B. Also called a *non-deductive implication*.
$A \leftrightarrow B$	*Contingent equivalence* between A and B. Also called a *bi-conditional*.
ß $=_{\mathrm{df}} S$	The symbol ß is *defined* by the statement S.
$A \stackrel{\mathrm{def}}{=} B$	A and B are *identical by definition*.
$A \sim B$	A and B are *equivalent*.
$A \nsim B$	A and B are not *equivalent*.
$\exists\ldots$	The *existential* quantifier ("‹there exists›\‹at least one of› something such that …").
$\forall\ldots$	The *universal* quantifier ("for all\each\‹every one› of something such that …").
$f{:}A \rightarrow B$	The *function* f from the domain A to the co-domain B.
$\mathbf{dom}\, f$	The *domain* of the function f.
$\mathbf{ran}\, f$	The *range* of the function f.
$\mathbf{cod}\, f$	The *co-domain* of the function f.
$f(x)$	The *image* of x under the function f.
$\mathbb{N}$	The set of *natural numbers* (includes the number 0).
$\mathbb{N}^{+}$	The set of positive *natural numbers* (excludes the number 0).
$\mathbb{R}$	The set of *real numbers*.
$\mathbb{Q}$	The set of *rational numbers*.
$\overline{\mathbb{Q}}$	The set of *irrational numbers*.

$\varnothing$	The *null set* (called the "empty set" by unprofessional mathematicians who still use the name taught to them in kindergarten).
$A \precsim B$	The set A *precedes* the set B.
$A \prec B$	The set A *strictly precedes* the set B.
$A \backslash B$	The *set difference* between sets A and B. Also called *relative complement* of B in A.
$\wp(A)$	The *power set* of the set A.
$A \subseteq B$	The set A is a *subset* of the set B.
$A \subset B$	The set A is a *proper subset* of the set B.
$\aleph_0$	The smallest (and only) *transfinite cardinal number* (the other *transfinite cardinals* are just smoke and mirrors).
$[0, 1]$	The *unit interval*.