The Collapse of Transfinite Cardinals

Revision 1

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Abstract

The purpose of this article is to prove that the existence of transfinite cardinals is untenable at this point in time because the traditional “proofs” of their existence are logically invalid. The only transfinite cardinal that may be admitted is $\aleph_0$. The definition of transfinite cardinals depends upon Cantor’s theorem and the non-denumerability of the real numbers. The “proofs” of those theorems depend upon Cantor’s diagonal method. I will prove that the diagonal method is self-contradictory, and consequently, the existence of transfinite cardinals is untenable (except for $\aleph_0$). I will not prove, with certainty, that the transfinite cardinals do *not* exist, but only that they have not yet been proven to exist.

Preface

In the year 1988, when I first encountered *transfinite cardinals*, I realised within a few minutes of reading the “proof” that the set of *real numbers* is *non-denumerable*, that the “proof” introduced two mutually contradictory assumptions. In the construction of a number belonging to the set of *real numbers*, I had noticed that there was no proof that such a number can exist under the given hypothesis that the *real numbers* are *denumerable*. In fact, it was quite obvious to me that the process of construction was inconsistent with that hypothesis — that such a number could not be constructed by its own definition. Apparently, no one has (publicly) given a rigorous proof of the error. I therefore offer this article to the mathematical community.

This article assumes that the reader has an elementary understanding of *transfinite cardinal numbers*. Such an understanding can be obtained from most books on set theory. The article would be of interest mainly to pure mathematicians.

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Main Article

Introduction

A *cardinal number*, in common mathematics, is the number of elements in a set. The definition of *cardinal number* was generalised by Georg Cantor (1845–1918) during his investigations into the foundations of set theory. The **cardinal number** (or **cardinality**) of a given set A (denoted $|A|$) is the family of sets that are *equivalent* to A. Two sets, A and B, are **equivalent** (denoted $A \sim B$) if one can be put into a one-to-one correspondence with the other. Some mathematicians define *cardinal number* in a different way, but all definitions of *cardinal number* are equivalent.

The *cardinal number* of a set whose elements can be counted is denoted by the number of elements in that set, that is to say, by a *natural number* or zero. The number of elements in some sets cannot be counted, and therefore cannot be denoted by a *natural number*. Such sets require new symbols to represent their *cardinality*. Those *cardinal numbers*, represented by the new symbols, are said to be **transfinite cardinals**. For example, the *cardinal number* of the set of *natural numbers*, $\mathbb{N}$, is denoted by $\aleph_0$ (i.e. $|\mathbb{N}| = \aleph_0$).

Georg Cantor had claimed to have proven that there is more than one *transfinite cardinal*. Today, mathematicians accept his (so-called) proof and conclusion. This article presents a rigorous proof that the claimed "proof" is invalid, and so, the conclusion that there is more than one *transfinite cardinal* cannot be accepted. Note well that this article does not conclude that there is only one *transfinite cardinal* and no other, but that there is no logical justification in concluding that there is more than one *transfinite cardinal*. The only *transfinite cardinal* that can be proven to exist is $\aleph_0$.

Appendix B contains a list of the definitions of the mathematical terms and symbols used in this article.

1. Logical Deduction

It will be seen, at a later part of this article, that the error in the argument that there is more than one *transfinite cardinal* is due to an inadequate understanding by mathematicians of the process of logical deduction. I will therefore give a simplified summary of the process in this section.

A logical deduction consists of given information (called the 'premise' or 'premises') that is true in some sense, and information that is not immediately apparent (called the 'conclusion'). The conclusion must <u>necessarily</u> be true in exactly the same sense that the premise is true. The only way that the conclusion can necessarily be true is for it to be intrinsic to the premise, that is to say, the conclusion must already have been communicated (but usually not in an obvious way) in the premise. Because, in most situations, the conclusion is not obviously apparent in the premise, a process (called 'deduction') is carried out to make the conclusion obvious.

The process of deduction involves a sequence of intermediate deductive steps where the conclusion of the final step is the conclusion sought. The conclusions of the intermediate steps and the original premises serve as the premises for the following steps. In a complex system of deductions, the initial premises are called 'axioms', and each major conclusion is called a ‘theorem’. The intermediate steps involved in producing a theorem is called a ‘proof’ (if those steps are produced correctly). An example of a complex system of deductions is pure mathematics, the axioms of which are the axioms of set theory. Most theorems in pure mathematics are therefore conclusions from those axioms.

Sometimes, errors are inadvertently introduced in the process of proving a theorem. If the process contains errors, the proof (properly called an ‘argument’ not ‘proof’) is said to be invalid. This does not mean that the theorem is necessarily false, but it does mean that the theorem is unreliable. The theorem may well happen to be true, but it could also be false; which one of the two cannot be determined from that erroneous argument.

A particular error, which is exposed in this article, involves the concept of existence of mathematical entities. There seems to be a misunderstanding among some mathematicians about the meaning of existence in mathematics. In mathematics, existence does not have the same meaning as it does in ordinary life or in philosophy. To say that an entity exists mathematically is not to say that the entity exists in some realm of reality, or has some kind of metaphysical existence. In mathematics, to say that an entity exists is to say that the entity is a member of some defined set. For example, to say that there exists x such that $3 + x = 5$ is to say that the set $\{x : 3 + x = 5\}$ is not the *null set*. The concept of mathematical existence does not involve any concept outside of mathematics itself; any claim that it does is a philosophical claim not a mathematical claim.

The assertion that an entity exists (whether the assertion is explicitly made or not) in a proof is a step of the proof, and it is required that the existence be deduced from the premises just as any other step is required to be deduced. For example, the statement “let $y = 2 + x$” in a proof actually means “(for each x) there exists y such that $y = 2 + x$”, which in turn means $\{y : y = 2 + x\} \neq \varnothing$ for each x. Typically, no explicit deduction is given in such simple cases since the existence is obvious; in our example, the existence can be deduced from the closure law of algebra (namely, if $a, b \in \mathbb{R}$ then $a + b \in \mathbb{R}$). If the asserted existence of an entity cannot be deduced from the premises (the axioms of set theory in the case of mathematics) then the argument is invalid. If the conditions used to define the entity are self-contradictory then it is impossible to deduce the existence of that entity from any premises at all, since nothing can satisfy a self-contradictory condition. For example, if a statement in a proof is “let a be a decimal number such that its third decimal position differs from itself”, then that statement cannot be deduced from any premises since the condition for the existence of a is self-contradictory, and so, a (the decimal number) cannot exist (mathematically speaking).

In a proof, existence of a mathematical entity can be expressed in a number of ways. Examples are: “Let x be …”, “Define Y such that …”, “Construct P …”. These are all statements of existence, and a proper proof should contain a deduction of the existence of the indicated entity. If the entities cannot be deduced to exist then an assumption has been made, and the “proof” becomes invalid. Of special concern is a statement like “Define Y to be …”. Mathematical entities cannot just be defined into existence. The statement should be replaced by “There exists Y such that …”. It needs to be made certain that the condition of existence is not self-contradictory. The whole system of *transfinite cardinals* (excluding $\aleph_0$) collapses because the “proofs” of two critical theorems contain a statement of existence with a condition that is self-contradictory, as shown in full explicit detail in this article.

Some theorems cannot be proven by direct deduction from their premises, so an indirect deduction is used called 'proof by contradiction'. The concept will be explained as follows. Suppose that theorem T needs to be proven, that is, deduced from the axioms. The following situations are possible:
(**a**) T is intrinsic to the axioms,
(**b**) $\sim T$ (the negation of T) is intrinsic to the axioms,
(**c**) neither T nor $\sim T$ is intrinsic to the axioms, or
(**d**) both T and $\sim T$ cannot be intrinsic to the axioms since that would make the axioms self-contradictory (it is this fact that proof by contradiction utilises).

If case (c) applies, then neither T nor $\sim T$ can be theorems, so that case will not be considered here. To prove T using proof by contradiction, $\sim T$ is hypothetically assumed to be true (i.e. intrinsic to the axioms). If a contradiction is deduced under that assumption then T must actually be intrinsic to the axioms, since a contradiction can exist only in case (d), with $\sim T$ being hypothetically but not actually intrinsic to the axioms.

The contradiction mentioned in the previous paragraph is deduced in the following manner. Some conclusion, A, is deduced from the axioms and the hypothetical $\sim T$. Then some other conclusion, B, is deduced from the axioms and $\sim T$ in such a way that A and B are contradictory. Each deduction must be a valid deduction in its own right with no assumptions made.

2. Denial and Refutation

Humans, including mathematicians, make mistakes. Many mathematical "proofs" in history have been shown to be invalid. It is therefore proper and necessary that potential proofs be inspected and corrected by other mathematicians. However, a correction of a proof may itself be invalid, so the correction needs to be just as rigorous a proof as any other proof needs to be. An argument is corrected by being refuted. NOTE WELL that a denial is not a refutation — there is a difference.

The following is a list of the requirements for a valid refutation of a logical argument.

(1) The actual statement in the argument that is in error must be explicitly identified. If no statement can be identified as an error then there may not be an error in the argument.

(2) The reasons that the statement is in error must be stated explicitly. Simply claiming that a statement is in error without stating the reason does not constitute a refutation.

(3) The stated reasons that a statement is in error must be precise. The reasons must be at least as precise as the argument. Vague and overgeneralised reasons are not reasons at all.

(4) The stated reasons that a statement is in error must be proven to be correct. If the reasons are themselves logically invalid then no refutation has been made.

The most common errors in logical arguments are made by the introduction of arbitrary assumptions that cannot be deduced from a relevant set of axioms, and ambiguous use of symbols, that is, when the same symbol is defined one way then used as if it were defined in a different way.

No doubt, many people will want to try to refute my arguments that, firstly, the traditional argument for Cantor's theorem is invalid, and secondly, that the traditional argument that $\mathbb{R}$ is *non-denumerable* is invalid. Any refutation will need to comply with the four rules mentioned above for it to be a valid refutation, otherwise it will just be a denial (which has no logical merit), and my arguments will remain unrefuted. In particular, an attempted refutation that is just a paraphrasing of existing "proofs" that Cantor was correct, as if such "proofs" contain no errors, will not be accepted as a valid refutation of my arguments, since no refutation will have been made at all. Also, any attempted refutation that explicitly or implicitly assumes that Cantor's theorem is true or that *real numbers* are *non-denumerable* will not be accepted as a valid refutation of my arguments, since those two assumptions themselves are at issue.

3. The Invalidity of the Proof of Transfinite Cardinals

The issue is this: are there any *cardinal numbers* greater than $\aleph_0$? The answer in traditional mathematics is "yes". This conclusion is traditionally justified by "proving", firstly, that $|A| < |\wp(A)|$ for every set A (finite or *infinite*), which, in particular, implies that $|\mathbb{N}| < |\wp(\mathbb{N})|$, which implies $\aleph_0 < |\wp(\mathbb{N})|$ (since $|\mathbb{N}| = \aleph_0$), and secondly that $|\mathbb{N}| < |\mathbb{R}|$, which implies $\aleph_0 < |\mathbb{R}|$. The whole system of *transfinite cardinals* rests on those two conclusions. The main purpose of this article is to prove that the traditional argument for the first conclusion, $|A| < |\wp(A)|$, is at best invalid and at worst false. If it is invalid then the conclusion has not been proven to be true; if it is false then the correct conclusion is $|A| \le |\wp(A)|$. In either case, no logical deduction has been made to conclude that there does exist an *infinite* set that has a *cardinality* greater than $\aleph_0$. This article also proves that the traditional argument for the second conclusion, $|\mathbb{N}| < |\mathbb{R}|$, is invalid. Therefore, no logical deduction has been made to conclude that $\mathbb{R}$ is *non-denumerable*. In short, this article proves that no *cardinal number* greater than $\aleph_0$ has yet been correctly proven to exist, therefore the whole system of *transfinite cardinal numbers* collapses because it is not logically justified.

The details showing that the (so-called) proofs of $|A| < |\wp(A)|$ and $|\mathbb{N}| < |\mathbb{R}|$ are not justified are presented in the following subsections. I use the word 'proposition' to indicate that the conclusion has not definitely been proven to be true, thus it may be false.

3.1 Cantor's Theorem

Cantor's theorem, and the traditional argument for it, are shown in Proposition 1. Proposition 2 is used in the argument of Proposition 1. Recall that if a function, *f*, such that $f{:}A \to B$ is one-to-one and onto, then the function is called a **bijection** and there is said to be a **one-to-one correspondence** between A and B. A and B are also said to be **equivalent**.

PROPOSITION 1 (Cantor's Theorem)

IF A is a set THEN $|A| < |\wp(A)|$.

Argument:

1. $|\varnothing| < |\wp(\varnothing)|$ ➧ since $|\varnothing| = 0$ and $|\wp(\varnothing)| = |\{\varnothing\}| = 1$.

2. Let f be any one-to-one function such that $f{:}A \to \wp(A)$ (where $A \neq \varnothing$).

3. $A \precsim \wp(A)$ ➧ since for each $x \in A$ there is $\{x\} \in \wp(A)$.

4. f is not onto ➧ from step 2 and Proposition 2.

5. $A \prec \wp(A)$ ➧ from steps 2, 4, and 3.

6. $|A| < |\wp(A)|$ ➧ from steps 1 and 5.

■

Proposition 1 is true if A is finite; the issue is whether it is true for all *infinite* sets. The truth of the proposition depends entirely on the truth of Proposition 2, which is part of the traditional argument for Cantor's theorem.

Proposition 2 claims that it is impossible that there exists a function $f{:}A \rightarrow \wp(A)$ that is onto. In other words, the claim is that A and $\wp(A)$ are not *equivalent*. And if they are not *equivalent* then they have different *cardinalities* as shown in Proposition 1.

PROPOSITION 2

IF f is a one-to-one function such that $f{:}A \rightarrow \wp(A)$ THEN f is not onto.

Argument:

1. Assume, hypothetically, that f is onto.
2. Construct B such that $B = \{x : x \in A, x \notin f(x)\}$ ➧ see below.
3. $B \subseteq A$ ➧ from step 2.
4. $B \in \wp(A)$ ➧ from step 3 and by definition of $\wp(A)$.
5. There exists x_0 such that $x_0 = f^{-1}(B)$ ➧ from the condition of Proposition 1 and steps 1 and 4.
6. $f(x_0) = B$ ➧ from step 5.
7. $x_0 \in A$ ➧ from step 5 and the condition of Proposition 1.
8. $x_0 \in B$ if and only if $x_0 \notin f(x_0)$ ➧ from steps 2 and 7.
9. $x_0 \in B$ if and only if $x_0 \notin B$ ➧ from steps 8 and 6.
10. step 9 is a contradiction.
11. f is not onto ➧ from steps 1 and 10.

■

The crucial step in Proposition 2 is step 2. Some mathematicians use the word "define" or "let" instead of "construct". Whichever fancy word is used, the statement is asserting the existence of a set, B, whose elements, x, satisfy the condition $x \in A$ and $x \notin f(x)$. As mentioned previously (see section **1. Logical Deduction**), the existence of B must be deduced from the axioms of set theory and the hypothetical condition at step 1, whether that existence itself is hypothetical or not. If the existence is not deduced, or if it is not or cannot be an axiom of set theory, then it is an arbitrary assumption. In that case the whole argument is not a genuine deduction and is therefore invalid.

The traditional argument for the justification of Cantor's theorem fails because step 2 is not, and in fact can not be, deduced from the axioms of set theory and the hypothetical condition at step 1. The reason that the set B at step 2 cannot be deduced is that the definition for that set, namely $x \in A$ and $x \notin f(x)$, is self-contradictory (taking into account the way that f is defined), and nothing can exist (in any way) that is self-contradictory.

The following Theorem 1 proves that the set B at step 2 of Proposition 2 cannot exist under any circumstance.

THEOREM 1

IF f is a one-to-one and onto function (a *bijection*) such that $f{:}A \rightarrow \wp(A)$ THEN there does not exist B such that $B = \{x : x \in A, x \notin f(x)\}$.

Proof:

1. Assume, hypothetically, that B exists.
2. $B \subseteq A$ ➧ from the definition of B in Theorem 1.
3. $B \in \wp(A)$ ➧ from step 2 and by definition of $\wp(A)$.
4. There exists x_0 such that $x_0 = f^{-1}(B)$ ➧ from the condition of Theorem 1 and step 3.
5. $f(x_0) = B$ ➧ from step 4.
6. $x_0 \in A$ ➧ from step 4 and the condition of Theorem 1.

7. $x_0 \in B$ if and only if $x_0 \notin f(x_0)$ ➧ from the definition of B in Theorem 1 and step 6.
8. $x_0 \in B$ if and only if $x_0 \notin B$ ➧ from steps 7 and 5.
9. Step 8 is a contradiction.
10. B is not a set and does not exist ➧ from steps 8, 1, and 9.
■

Theorem 1 is the theorem that proves that the traditional argument for Cantor's theorem is invalid. Mathematicians, having the same prejudices and biases as ordinary people, would no doubt very much like to refute Theorem 1, but are likely, instead, to just deny it (see section **2. Denial and Refutation**). The reader should compare Theorem 1 with Proposition 2. The theorem and proposition are almost identical in form; the main difference being that the proposition has an arbitrary assumption (step 2). So, if the argument of Proposition 2 is accepted as valid by mathematicians, then why not the proof of Theorem 1 (which does not contain an arbitrary assumption)?

Those mathematicians who have an understanding of axiomatic set theory (Zermelo-Fraenkel) would probably argue as follows.

> Theorem 1 is in fact true. But its truth only shows that if the function f were intrinsic to the axioms of set theory, then set B ought to be an axiom of set theory as an instance of the axiom schema of separation ($\exists B \forall x\ (x \in B \leftrightarrow x \in A \wedge p(x)$). However, the set B is $x \in B \leftrightarrow x \in A \wedge x \notin f(x)$ as an axiom, but implies $x_0 \in B \leftrightarrow x_0 \notin B$ (as shown at step 8 of Theorem 1) since $x_0 \in A$ is true. Thus, there would be a self-contradictory axiom of set theory, and so f cannot be intrinsic to the axioms of set theory. That is to say, f cannot be a *bijection* and so $|A| < |\wp(A)|$ is true.

Such an argument is flawed, as will be seen when the situation is analysed as follows. Consider the following hypothetical axioms of set theory.

(1a) $\exists f_b \exists A$ (f_b is a *bijection* and $f_b{:}A \rightarrow \wp(A)$ and A is an *infinite* set)

(1b) $\sim\exists f_b \exists A$ (f_b is a *bijection* and $f_b{:}A \rightarrow \wp(A)$ and A is an *infinite* set)

(1b) $\forall f_n \forall A$ (f_n is not a *bijection* and $f_n{:}A \rightarrow \wp(A)$ and A is an *infinite* set)

(2) $\exists B_b \forall x\ (x \in B_b \leftrightarrow x \in A \wedge x \notin f_b(x))$ [where f_b is a *bijection* and $f_b{:}A \rightarrow \wp(A)$ and A is an *infinite* set]

(3) $\exists f_n \exists A$ (f_n is not a *bijection* and $f_n{:}A \rightarrow \wp(A)$ and A is an *infinite* set)

(4) $\exists B_n \forall x\ (x \in B_n \leftrightarrow x \in A \wedge x \notin f_n(x))$ [where f_n is not a *bijection* and $f_n{:}A \rightarrow \wp(A)$ and A is an *infinite* set]

Note that (1b) is the negation of (1a), and the two (1b) are logically equivalent. The logical possibilities of the above hypothetical axioms are:

(a) (1a) is intrinsic to the axioms of set theory (and therefore (1b) is not).

(b) (1b) is intrinsic to the axioms of set theory (and therefore (1a) is not).

(c) neither (1a) nor (1b) is intrinsic to the axioms of set theory.

(d) (3) is intrinsic to the axioms of set theory (this can be proven).

(e) (4) is intrinsic to the axioms of set theory (this can be proven).

The implications of the above are:

(f) If (a) applies then (2) is self-contradictory (see Theorem 1), and therefore set theory is inconsistent (the axioms contain at least one contradiction).

(g) If (b) applies then (2) cannot be defined (since its definition depends on (1a)), and therefore (2) cannot be an axiom.

(h) If (c) applies then (2) cannot be defined (since its definition depends on (1a)), and therefore (2) cannot be an axiom.

Note that (2) can never be an axiom of set theory under any circumstance (hypothetical or otherwise) and therefore can never be a valid step in any proof. That is why step 2 of Proposition 2 (and therefore the whole argument of that proposition) is invalid.

Considering the above possibilities and implications; there are three possible conclusions:

(i) Statement (a) applies and Cantor's theorem is false and set theory is inconsistent (no one has yet proven that set theory is consistent).

(j) Statement (b) applies and Cantor's theorem is true but results in other contradictions in set theory (usually called "paradoxes").

(k) Statement (c) applies and neither Cantor's theorem nor its negation can be proven (i.e. Cantor's theorem is 'undecidable').

The traditional argument for Cantor's theorem only proves statement (f) — it does not prove the conclusion (j). There is still the possibility that Cantor's theorem is false and that set theory is inconsistent, or that Cantor's theorem is undecidable. In other words, the argument for Cantor's theorem contains an arbitrary introduction of a set deliberately defined to be self-contradictory (step 2 of Proposition 2), and so the correct conclusion is that if the function f is a *bijection* (step 1 of Proposition 2) then set theory is inconsistent, rather than the claimed conclusion that the function f cannot be a *bijection*. Note that even if statement (c), and therefore statement (k), is rejected, there still remains the other two possible conclusions.

Some mathematicians try to prove Cantor's theorem by first defining $B = \{x : x \in A, x \notin f(x)\}$ where f is any function $f{:}A \to \wp(A)$, *bijective* or not. They then hypothetically assume that f is *bijective*, which results in a contradiction, then conclude that f cannot be *bijective*. However, the proper analysis of such an argument is as follows. B is either B_n or B_b depending on, respectively, whether f_n (if f is not a *bijection*) or f_b (if f is a *bijection*) is being considered. But, as shown in this article (see Theorem 1), B_b cannot exist, therefore the only option left is that B is B_n (which means that f is f_n). So, therefore, such an argument never considers f_b, which means that f_b can still exist. In other words, the definition of B restricts the function f only to those that are not *bijective*; *bijective* functions (if any) are never considered. Cantor's theorem is therefore not justified.

The following Proposition 3 implies that Cantor's theorem is actually false. I have made it a proposition because there is a particular line in the argument that may be questioned by some mathematicians.

PROPOSITION 3

There exists A such that $|A| = |\wp(A)|$.

Argument:

1. There exists A such that $A = \wp(A) \cup X$ where $X \neq \varnothing$ and $X \neq A$ ➧ see below.

2. There exists $f{:}A \to \wp(A)$ defined by $f(x) = \{x\}$ ➧ since for each $x \in A$ there is $\{x\} \in \wp(A)$.

3. There exists $g{:}\wp(A) \to A$ defined by $g(x) = x$ ➧ since $\wp(A) \subseteq A$.

4. $A \precsim \wp(A)$ ➧ from step 2, and since f is one-to-one and into.

5. $|A| \leq |\wp(A)|$ ➧ from step 4.
6. $\wp(A) \precsim A$ ➧ from step 3, and since g is one-to-one and into.
7. $|\wp(A)| \leq |A|$ ➧ from step 6.
8. $|A| = |\wp(A)|$ ➧ from steps 5 and 7 and the Schroder-Bernstein theorem.
■

Proposition 3 shows that Cantor's theorem is false because the conclusion shows an exception to the theorem. Some mathematicians will argue that the set A at step 1 cannot exist, and therefore the argument is invalid. However, consider the following. The members of A contain at least the members of X (where X is any set as specified at step 1). If $x \in A$ then $\{x\} \in \wp(A)$ therefore $\{x\} \in A$; if $x_1, \ldots, x_n \in A$ then $\{x_1, \ldots, x_n\} \in \wp(A)$ therefore $\{x_1, \ldots, x_n\} \in A$; if $x_1, \ldots \in A$ then $\{x_1, \ldots\} \in \wp(A)$ therefore $\{x_1, \ldots\} \in A$. The members of A are well defined, so it seems that there is no reason to reject step 1 of Proposition 3.

3.2 Non-denumerability of ℝ

The traditional argument that ℝ is *non-denumerable* is shown in Proposition 4. Proposition 5 is used in the argument of Proposition 4. To make the arguments more precise, I will define a sequence of single digits in Definition 1. A set of such sequences will later correspond to the *real numbers*.

DEFINITION 1

IF D is a sequence defined by $D = \langle D(1), D(2), \ldots\rangle$ where $D(i) \in \{0, 1, \ldots, 9\}$ is a term of D $[i = 1, 2, \ldots]$, and there does not exist a non-zero term of the sequence where all subsequent terms are zero THEN D is said to be a **digit sequence**.

PROPOSITION 4 (Non-denumerability of ℝ)

ℝ is *non-denumerable*.

Argument:

1. Let $S = \{x : x$ is a *digit sequence*$\}$.
2. $\mathbb{R} \sim \{x : 0 < x < 1\} \sim S$ ➧ from set theory.
3. Let f be any one-to-one function such that $f{:}\mathbb{N} \to S$.
4. f is not onto ➧ from steps 3 and 1 and Proposition 5.
5. S is *non-denumerable* ➧ from steps 3 and 4.
6. ℝ is *non-denumerable* ➧ from steps 2 and 5.
■

The truth of the above proposition depends entirely on the truth of Proposition 5, which is part of the traditional argument for the non-denumerability of ℝ.

Proposition 5 claims that it is impossible that there exists a function $f{:}\mathbb{N} \to S$ that is onto ($S = \{x : x$ is a *digit sequence*$\}$). In other words, the claim is that ℕ and S are not *equivalent*, and since S is *equivalent* to ℝ, then ℕ and ℝ are not *equivalent*. And if they are not *equivalent* then they have different *cardinalities* as implied in Proposition 4.

PROPOSITION 5

IF f is a one-to-one function such that $f{:}\mathbb{N} \to S$ where $S = \{x : x$ is a *digit sequence*$\}$ THEN f is not onto.

Argument:

1. Assume, hypothetically, that f is onto.
2. Construct T such that $T \in S$ and $T(i) \neq f(i)(i)$ $[i = 1, 2, \ldots]$ ➧ see below.

3. $T \in S$ ➧ from step 2.
4. There exists n such that $n = f^{-1}(T)$ ➧ from the condition of Proposition 5 and steps 1 and 3.
5. $f(n) = T$ ➧ from step 4.
6. $T(n) \neq f(n)(n)$ ➧ from step 2.
7. $T(n) \neq T(n)$ ➧ from steps 5 and 6.
8. step 7 is a contradiction.
9. f is not onto ➧ from steps 8 and 1.
■

The crucial step in Proposition 5 is step 2. Some mathematicians use the word "define" or "let" instead of "construct". Whichever fancy word is used, the statement is asserting the existence of a *digit sequence* T, that satisfies the condition $T \in S$ and $T(i) \neq f(i)(i)$ $[i = 1, 2, \ldots]$. T is such that its i^{th} digit differs from the i^{th} digit of the sequence indicated by $f(i)$. This is an instance of the familiar Cantor's diagonal method expressed in symbolic form. As mentioned previously (see section **1. Logical Deduction**), the existence of T must be deduced from the axioms of set theory and the hypothetical condition at step 1, whether that existence itself is hypothetical or not. If the existence is not deduced, or if it is not or cannot be an axiom of set theory, then it is an arbitrary assumption. In that case the whole argument is not a genuine deduction and is therefore invalid.

The traditional argument for the justification that $\mathbb{R}$ is *non-denumerable* fails because step 2 is not, and in fact can not be, deduced from the axioms of set theory and the hypothetical condition at step 1. The reason that the *digit sequence*, T, at step 2 cannot be deduced is that the definition for that sequence, namely $T \in S$ and $T(i) \neq f(i)(i)$ $[i = 1, 2, \ldots]$, is self-contradictory (taking into account the way that f is defined), and nothing can exist (in any way) that is self-contradictory. To illustrate the contradiction in attempting to construct T, consider the following. Suppose that we define an algorithm to construct a *digit sequence* such that its 10^{th} digit differs from itself, and all other digits are arbitrary. Mathematically speaking, if the digit sequence is D then $D(10) \neq D(10)$ and $D(i) \in \{0, 1, \ldots, 9\}$ $[i \in \mathbb{N}$ and $i \neq 10]$. Would it be possible to construct such a *digit sequence*? Would the algorithm be self-contradictory? The *digit sequence*, T, at step 2 of Proposition 5 is effectively like D. By virtue of the hypothesis that f is onto, T must be one of the $f(i)$ for some particular value, n, of the values of i (see step 4 of Proposition 5). Therefore $f(n) = T$ (for example, n could be 10). But, by the definition of T, the digit $T(n)$ differs from $f(n)(n)$, that is to say that $T(n)$ differs from $T(n)$. Mathematically speaking, $T(n) \neq T(n)$. Therefore, the construction of T is just as impossible as the construction of D shown above. The contradiction in Proposition 5 results from the way that T is defined, rather than from the hypothetically assumption that f is onto.

The following Theorem 2 proves that the *digit sequence*, T, at step 2 of Proposition 5 cannot exist under any circumstance.

THEOREM 2

IF f is a one-to-one and onto function such that $f{:}\mathbb{N} \rightarrow S$ where $S = \{x : x$ is a *digit sequence*$\}$ THEN there does not exist T such that $T \in S$ and $T(i) \neq f(i)(i)$ $[i = 1, 2, \ldots]$.
Proof:
1. Assume, hypothetically, that T exists.
2. $T \in S$ ➧ from step 1 and the definition of T in Theorem 2.
3. There exists n such that $n = f^{-1}(T)$ ➧ from step 2 and the condition of Theorem 2.
4. $f(n) = T$ ➧ from step 3.

5. $T(n) \neq f(n)(n)$ ➧ from the definition of T in Theorem 2.
6. $T(n) \neq T(n)$ ➧ from steps 4 and 5.
7. step 6 is a contradiction.
8. T does not exist ➧ from steps 7 and 1.
■

Theorem 2 is the theorem that proves that the traditional argument for the *non-denumerability* of $\mathbb{R}$ is invalid. As for Theorem 1, mathematicians wanting to refute Theorem 2 are likely, instead, to just deny it (see section **2. Denial and Refutation**). The reader should compare Theorem 2 with Proposition 5. The theorem and proposition are almost identical in form; the main difference being that the proposition has an arbitrary assumption (step 2). So, if the argument of Proposition 5 is accepted as valid by mathematicians, then why not the proof of Theorem 2 (which does not contain an arbitrary assumption)?

Another "proof" that $\mathbb{R}$ is *non-denumerable*, which will not be shown in detail here, involves constructing an infinite sequence of *closed intervals* on the interval [0, 1], each interval being a *proper subset* of the previous one. The intervals are constructed in such a way that the n^{th} interval excludes the member of [0, 1] corresponding to $n \in \mathbb{N}$ (it is assumed hypothetically that [0, 1] is *denumerable*). The result being that, by another theorem, there should exist a member of [0, 1] that belongs to all the intervals, but that due to the way the intervals are constructed, no such member exists – thus a contradiction, and therefore $\mathbb{R}$ is *non-denumerable*.

The "proof" is invalid on two accounts. Firstly, as in the other "proofs", either such a sequence of intervals cannot exist and, therefore, cannot be constructed, or the other theorem (that there exists a member of [0, 1] that belongs to all the intervals) is false. Secondly, if all the intervals, including the interval [0, 1], were limited only to rational numbers, then, if the argument is correct, the conclusion would be that the rational numbers are *non-denumerable*. This conclusion is incorrect, so the argument must be invalid.

Some authors have even tried to make use of the (false) theorem that for any *denumerable* set A such that $A \subseteq \mathbb{R}$, there exists x such that $x \in \mathbb{R}$ and $x \notin A$ (that is, $A \subset \mathbb{R}$). This theorem is immediately false, since if $\mathbb{R}$ is *denumerable* then $\mathbb{R} \subseteq \mathbb{R}$ implies that there exists x such that $x \in \mathbb{R}$ and $x \notin \mathbb{R}$ — a contradiction.

4. Conclusion

Currently, mathematicians believe that they have a valid proof that the following two propositions are true.

(a) For each set A, finite or *infinite*, then $|A| < |\wp(A)|$.

(b) $\mathbb{R}$ is *non-denumerable*, meaning that $|\mathbb{N}| < |\mathbb{R}|$. Since $\aleph_0$ is defined to be $|\mathbb{N}|$, then $\aleph_0 < |\mathbb{R}|$.

The conclusion from the above two propositions, if they are true, is that there is a sequence of increasing *cardinal numbers* greater than $\aleph_0$, defined from proposition (a) as follows: $|\mathbb{N}| < |\wp(\mathbb{N})| < |\wp(\wp(\mathbb{N}))| < |\wp(\wp(\wp(\mathbb{N})))| < \ldots$, which are designated, respectively, by the symbols $\aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots$. These are the *transfinite cardinal numbers*. It can be proven that $|\mathbb{R}| = |\wp(\mathbb{N})| = \aleph_1$. A question that can be asked is whether there exists x such that $\aleph_0 < x < \aleph_1$. The hypothesis that there is not such an x is called the ‘continuum hypothesis’. The truth or falsity the continuum hypothesis is agreed by mathematicians to be undecidable, that is to say that neither the hypothesis nor its negation can be determined from the axioms of set theory.

In this article, I have shown that the traditional arguments for both propositions (a) and (b) are logically invalid. This implies that they are inconclusive, and therefore, propositions (a) and (b) cannot be maintained to be necessarily true. One or both of them may be true, but as yet no one has given a valid proof that they are. Without those propositions being definitely true, the sequence of *transfinite cardinal numbers*, excluding $\aleph_0$, would be unjustified. The continuum hypothesis would be meaningless, since it would not be defined.

I have also given a strong argument supporting the proposition that there exists at least one *infinite* set, A, such that $|A| = |\wp(A)|$. If that proposition is true, then proposition (a) would be false, and *transfinite cardinal numbers* would have to be regarded as pure mathematical fictions.

It should be noted that if proposition (a) is true then certain other contradictions (and undesirable consequences), not mentioned in this article, result. Those embarrassing contradictions are called “paradoxes” by mathematicians. Some mathematicians create fudges in set theory in an attempt to cover up their embarrassment. If proposition (a) is changed to $|A| \leq |\wp(A)|$, then those “paradoxes” would not arise and no fudges would need to be created.

Appendix A

This Appendix A shows the critical proposition that makes or breaks the whole system of *transfinite cardinals* greater than $\aleph_0$. The existence or non-existence of *transfinite cardinals* depends on whether the proposition is true or false. If true, *transfinite cardinals* greater than $\aleph_0$ do not exist; if false, the question is unanswered.

The following two notations are used in this appendix.

A^∞ denotes the infinite product set of A. $A^\infty = A \times A \times \ldots = \prod_1^\infty A$

$\wp^{\underline{n}}(X)$ denotes n recursive power sets. $\wp^{\underline{n}}(X) = (\wp \circ \wp \circ \ldots \circ \wp)(X) = \wp(\ldots(\wp(X))\ldots)$ (where the number of $\wp$'s is n).

Proposition A1

$|\mathbb{N}^\infty| = \aleph_0$.

Argument:

1. $\lim_{n\to\infty} |\mathbb{N}^n| = \aleph_0$ $[n \in \mathbb{N}^+]$ ➧ since $|\mathbb{N}^n| = \aleph_0{}^n = \aleph_0$ from set theory.

2. $|\mathbb{N}^\infty| = \aleph_0$ ➧ from step 1 (see below).

■

Step 2 of Proposition A1 (and therefore Proposition A1 itself) is not conclusive, and so the argument for the proposition is not a proof. However, consider that, in general, when the partial sums of an infinite series tend towards a particular value, the sum of the infinite series is defined to be equal to that particular value. For example, if $\lim_{n\to\infty} \sum_{i=1}^n u_i = a$, then, by definition, $\sum_{i=1}^\infty u_i = a$. The argument of Proposition A1 is consistent with the example. If the proposition is true, the following theorems follow.

Theorem A1

IF A is an *infinite* set THEN $\aleph_0 \le |A|$.

Proof:

1. There exists X such that $X \sim A$ ➧ from the definition of an *infinite* set.

2. $X \subset A$ ➧ from the definition of an *infinite* set.

3. $|X| = |A|$ ➧ from step 1.

4. Either $|X| < \aleph_0$ or $\aleph_0 \le |X|$ ➧ from the Law of Trichotomy.

5. Assume, hypothetically, that $|X| < \aleph_0$. ➧ from step 4.

6. $|X| \in \mathbb{N}$ ➧ from step 5.

7. $|X| \ne |A|$ ➧ from steps 6 and 2 and the condition of Theorem A1.

8. Steps 3 and 7 are contradictory.

9. $\aleph_0 \le |X|$ ➧ from steps 8, 4, and 5.

10. $\aleph_0 \le |A|$ ➧ from steps 9 and 3.

■

The next theorem establishes the condition under which $|\mathbb{R}| = \aleph_0$. Under that condition, the theorem concludes that $\mathbb{R}$ is *denumerable*.

THEOREM A2

IF $|\mathbb{N}^\infty| = \aleph_0$ THEN $|\mathbb{R}| = \aleph_0$.

Proof:

1. There exists A such that $A = \{x : x$ is a *digit sequence*$\}$ ➧ from set theory.
2. $A \subset \mathbb{N}^\infty$ ➧ from set theory.
3. $A \precsim \mathbb{N}^\infty$ ➧ from step 2.
4. $|A| \leq |\mathbb{N}^\infty|$ ➧ from step 3.
5. $|A| \leq \aleph_0$ ➧ from the condition of Theorem A2 and step 4.
6. $\aleph_0 \leq |A|$ ➧ from Theorem A1.
7. $|A| = \aleph_0$ ➧ from steps 5 and 6.
8. $A \sim \{x : 0 < x < 1\} \sim \mathbb{R}$ ➧ from set theory.
9. $|\mathbb{R}| = \aleph_0$ ➧ from steps 8 and 7.
■

The next three theorems show the conditions under which Cantor's theorem ($|A| < |\wp(A)|$) would be false.

THEOREM A3

IF $|A| = \aleph_0$ and $|\mathbb{N}^\infty| = \aleph_0$ THEN $|A| = |\wp^n(A)|$ [for each $n \in \mathbb{N}^+$].

Proof:

1. $|\wp(\mathbb{N})| = |\mathbb{R}|$ ➧ from set theory.
2. $|\mathbb{R}| = \aleph_0$ ➧ from the second condition of Theorem A3, and Theorem A2.
3. $|\wp(\mathbb{N})| = \aleph_0$ ➧ from steps 1 and 2.
4. $|\mathbb{N}| = \aleph_0$ ➧ from set theory.
5. $|A| = |\mathbb{N}|$ ➧ from step 4 and the first condition of Theorem A3.
6. $\wp(A) \sim \wp(\mathbb{N})$ ➧ from step 5.
7. $|\wp(A)| = |\wp(\mathbb{N})|$ ➧ from step 6.
8. $|\wp(A)| = \aleph_0$ ➧ from steps 7 and 3.
9. $|A| = |\wp(A)|$ ➧ from steps 5, 4, and 8.
10. $|A| = |\wp^n(A)|$ [for each $n \in \mathbb{N}^+$] ➧ from recursive application of step 9, and from step 8.
■

THEOREM A4

IF A is an *infinite* set and $|\mathbb{N}^\infty| = \aleph_0$ THEN $|A| = \aleph_0$.

Proof:

1. $\aleph_0 \leq |A|$ ➧ from the first condition of Theorem A4, and Theorem A1.
2. if $\aleph_0 < |A|$ then there exists n such that $|A| = |\wp^n(\mathbb{N})|$ ➧ by definition of *transfinite cardinal numbers*.
3. $|\mathbb{N}| = \aleph_0$ ➧ by definition.
4. $|\mathbb{N}| = |\wp^n(\mathbb{N})|$ [for each $n \in \mathbb{N}^+$] ➧ by step 3, the second condition of Theorem A4, and Theorem A3.
5. $|A| = \aleph_0$ ➧ by steps 3, 4, and 2.
■

THEOREM A5

IF A is an *infinite* set and $|\mathbb{N}^\infty| = \aleph_0$ THEN $|A| = |\wp^n(A)| = \aleph_0$ [for each $n \in \mathbb{N}^+$].

Proof:

1. $|A| = \aleph_0$ ➧ from the condition of Theorem A5 and Theorem A4.
2. $|A| = |\wp^n(A)|$ [for each $n \in \mathbb{N}^+$] ➧ from step 1, the second condition of Theorem A5, and Theorem A3.
3. $|A| = |\wp^n(A)| = \aleph_0$ [for each $n \in \mathbb{N}^+$] ➧ from steps 2 and 1.
■

Theorem A5 shows that, assuming $|\mathbb{N}^{\infty}| = \aleph_0$, the *cardinality* of each *infinite* set would be $\aleph_0$. In particular, the *cardinality* of all recursive *power sets* of each *infinite* set would be $\aleph_0$, which would imply the death of all *transfinite cardinal numbers* greater than $\aleph_0$.

Appendix B

The following is a list of the definitions of the mathematical terms and symbols used in this article. A term being defined is shown in **this type**; an already defined term that is referenced in this article is shown in *this type*.

bijection

A function, f, such that $f{:}A \rightarrow B$ is one-to-one and onto.

cardinal number $|A|$

The family of sets that are *equivalent* to a given set, A, denoted by $|A|$. There are other equivalent ways of defining *cardinal number*. The *cardinal number* of the set of *natural numbers*, $\mathbb{N}$, is denoted by $\aleph_0$ (i.e. $|\mathbb{N}| = \aleph_0$).

cardinality

The *cardinal number* of a set.

closed interval $[a, b]$

The set of *real numbers* between a and b inclusive, $\{x : a \leq x \leq b\}$, denoted by $[a, b]$.

denumerable

A finite set or a set that is *equivalent* to $\mathbb{N}$.

digit sequence

See Definition 1 under **3.2 Non-denumerability of $\mathbb{R}$**.

equivalent (sets) $A \sim B$

The fact that two sets, A and B, can be put into a one-to-one correspondence with one another, denoted by $A \sim B$. Two sets, A and B, that are not *equivalent* are denoted by $A \nsim B$.

infinite (sets)

The set A is said to be *infinite* if there exists X such that $X \subset A$ and $X \sim A$.

natural numbers $\mathbb{N}$

The set of positive whole numbers together with the number 0, denoted by $\mathbb{N}$, or without the number zero, denoted by $\mathbb{N}^+$.

non-denumerable

An *infinite* set that is not *equivalent* to $\mathbb{N}$.

null set $\varnothing$

The set containing no members, denoted by $\varnothing$. The *null set* is also known as the 'empty set'.

open interval (a, b)

The set of *real numbers* between a and b exclusive, $\{x : a < x < b\}$, denoted by (a, b).

power set $\wp(A)$

The set of all *subsets* of a set, A, denoted by $\wp(A)$.

precede (sets) $A \precsim B$

A set A is said to *precede* a set B if there exists X such that $X \subseteq B$ and $X \sim A$. $A \precsim B$ denotes that *A precedes B*. If $A \precsim B$ then, by definition, $|A| \leq |B|$. $A \prec B$ means $A \precsim B$ and $A \nsim B$. If $A \prec B$ then, by definition, $|A| < |B|$.

proper subset $A \subset B$

The set, A, whose entire members are also members of a set, B, where A is not equal to B, denoted by $A \subset B$.

real numbers $\mathbb{R}$

The union of the set of rational and irrational numbers, denoted by $\mathbb{R}$.

subset $A \subseteq B$

The set, A, whose entire members are also members of a set, B, denoted by $A \subseteq B$.

transfinite cardinal

Another name for a *transfinite cardinal number*.

transfinite cardinal number

The set of *cardinal numbers* that are greater than or equal to $\aleph_0$, $\{x : \aleph_0 \leq x\}$.